Introduzione alla teoria della dimostrazione II. Qualche applicazione*

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Contents

Abstract

We apply cut elimination techniques coupled with majorization arguments to the study of the computational content of (fragments of) elementary arithmetic. We prove a classical result of Parsons, Mints and Takeuti identifying the primitive recursive functions as those provably recursive in arithmetic restricted to Σ_1-conditions. The result can be adapted to classify the content of bounded systems whose provable recursive functions are the PTIME-functions.

In second section we deal with higher order systems. We prove the Hauptsatz for *predicative comprehension* in full second order classical logic. We then consider the problem raised by the Russell paradox: we present a solution of the paradox, due to Grišin, which consists in working in a non-classical logic where the contraction rule is no more admissible, but which retains naive (hence impredicative) comprehension in full. The system still enjoys cut elimination.

Prerequisites: basic notions of classical recursion theory (primitive recursive functions, Turing Machines, rudiments of complexity theory). Natural deduction; first order arithmetic PA. Naive set theory and the issue of paradoxes; predicative vs. impredicative definitions.

*__Disclaimer__: le note che seguono costituiscono solo un hand-out preparato ad ausilio degli studenti; non sono fatte per essere diffuse e sono da correggere in tutti sensi !!. Si tratta solo di appunti di un corso breve, L'autore sarà comunque grato per ogni segnalazione di errori, commenti critici, etc.. . . .

1 Hauptsatz and computational content: the PMT theorem and its sharpenings

We describe a Tait-calculus for a language of arithmetic. In order to deal with (bounded) complexity sensible systems, we choose the algebra of binary strings as standard interpretation, following Ferreira (alternatives in the literature: the Buss formalism, refining Peano arithmetic, and Zambella's second order version of it).

1.1 Formal preliminaries

The arithmetical language of PA comprises:
(i) countably many individual variables $x_1, x_2, x_3, \ldots$;
(ii) the logical constants $\neg, \wedge, \vee, \forall, \exists$;
(iii) the predicates $=, <$ and an individual constant for the number zero (we use 0 as well); function symbols for primitive recursive functions.

Terms are inductively defined, as usual. We use x, y, z, u, v, w as metavariables for variables; t, t', t'', s, s', r, r' range over terms. $(t+1), t \cdot s, t+s$ stand for the successor of t, and the addition and product of t, s. The collection of numerals is the least collection of closed terms including 0 and closed under application of the successor symbol. The numeral representing the number k is denoted by $\overline{k}$.

Positive (negative) e-atoms have the form $t < s$, $t = s$ ($\neg t \leq s, \neg t = s$); an *e-atom* is either a positive or a negative e-atom. As usual, negation $\neg$ is extended to arbitrary formulas by letting $\neg\neg A = A$ (A positive atom); $\neg(A \wedge B) = (\neg A \vee \neg B)$; $\neg(A \vee B) = (\neg A \wedge \neg B)$; $\neg\forall x A = \exists x \neg A$. $A \to B$, $A \leftrightarrow B$ stand for their classical counterparts.
{That our official formulas are in negation normal form is a matter of convenience, due to the use of a Tait sequent calculus in a subsequent section}.

We remind that an *arithmetical formula* A is Δ_0 iff A is inductively generated from e-atoms by means of $\wedge, \vee$ and bounded quantifiers (this means that $A \equiv \forall x(x < t \to B)$ or $A \equiv \exists x(x < t \wedge B)$, $x \notin \mathrm{FV}(t), B \in \Delta_0$).

An arithmetical formula A is Σ_1 (Π_1) iff $A \equiv \exists x B\, (A \equiv \forall x B)$, for some $B \in \Delta_0$.

The class of Σ (Π)-formulas is the least class containing the Δ_0-formulas, and closed under $\vee$,$\wedge$, bounded quantifiers and existential (universal) quantifiers.

We restrict ourselves to the system, which comprises classical predicate logic with equality, defining equations for primitive recursive symbols, the induction principle for Σ_1-formulas and standard ordering axioms for $<$.

More precisely, we identify it with a classical finitary Tait calculus, where sequents of the form Γ are derived, Γ being a (finite) set of arithmetical formulas.

In particular a Q-axiom is a finite set of e-atoms having one of the following forms:

- E where E is a defining equation of a primitive recursive function (e.g. $t + 0 = t$, or $t \cdot (s+1) = t \cdot s + t$) or has the form
- $\neg(t+1) = 0$;

- $\neg(t+1) = (t+1), t = s$;
- $t = 0, t = pd(t) + 1$, where pd is the predecessor function symbol;
- $\neg t < s, (s-(t+1))+1+t = s$; here $-$ is the function symbol for truncated difference;
- $\forall z(\neg(z+1) + t = s), t < s$;

Definition 1.1. $I\Sigma_1$ consists of

1. the axioms (or initial sequents)

   (Lg) $\Gamma, \neg t = s, \neg A[x := t]$, $A[x := s]$, where A is an atomic $\mathcal{L}_d$-formula;

   (I1) $\Gamma, t = t$;

   (I2) $\Gamma, \neg t = s, \neg A[x := t]$, $A[x := s]$, (A e-atom);

   (At) Γ, Λ, where Λ is either a defining equation for a primitive recursive functions or else a Q-axiom;

2. standard logical rules for introducing $\wedge$, $\vee$, the quantifiers, and the cut rule:

   ($\wedge$)
   $$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma,\ A \wedge B}$$

   ($\vee$)
   $$\frac{\Gamma, A, B}{\Gamma,\ A \vee B}$$

   ($\forall$) if $a \notin FV(\Gamma)$,
   $$\frac{\Gamma, A[x := a]}{\Gamma,\ \forall x A}$$

   ($\exists$)
   $$\frac{\Gamma, A[x := t]}{\Gamma,\ \exists x A}$$

   (Cut)
   $$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

3. the induction rule for Σ_1-formulas: if A is Σ_1 and $a \notin FV(\Gamma)$,
   $$\frac{\Gamma, A(0) \qquad \Gamma, \forall x(A(x) \to A(x+1))}{\Gamma, A(a)}$$

The term t occuring explicitly in the premiss of the existential introduction is called *witnessing term of the inference*.

Remark 1.2. Every active formula in $I\Sigma_1$ is either Σ_1 or Π_1.

This suggests a relativized notion of *1-rank*; by induction on A, we define

- $rk_1(A) = 0$ se A is Σ_1 or Π_1; else

- $rk_1(\neg A) = rk_1(A) + 1$;
- $rk_1(A \circ B) = max(rk_1(A), rk_1(B)) + 1$ ($\circ$ connettivo binario);
- $rk_1(QxA) = rk_1(A) + 1$ (Q quantificatore)

We simply Let us call a derivation *quasi-normal* if its cut formulas are either Σ_1 or Π_1. By a straightforward adaption, we can define the relations:

1. $\mathcal{D} \vdash_k^m \Gamma$ sse $\mathcal{D}$ is a derivation in $I\Sigma_1$ of Γ such that:
   - every branch of $\mathcal{D}$ has at most m nodes;
   - every cut formula occurring in $\mathcal{D}$ has 1-rank $< k$
2. $\vdash_k^m \Gamma$ means that, for some $\mathcal{D}$, $\mathcal{D} \vdash_k^m \Gamma$

Theorem 1.3. *Every $I\Sigma_1$-derivation can be transformed into a quasi-normal $I\Sigma_1$-derivation of the same conclusion.*

Esercise 1.4. Prove the result in details: give the appropriate formulation of the inversion rules and the reduction lemma with length for the Tait system.

1.2 Majorizing $I\Sigma_1$-derivations

Definition 1.5. Let $\Gamma_{\mathbf{a}}$ be a set of formulas with free variables occurring in the list $\mathbf{a}$.
$\Gamma_{\vec{n}}[m, p]$ is the set obtained from $\Gamma_{\mathbf{a}}$ by

1. replacing each a_i by n_i ($n := n_1, \ldots, n_j$);
2. replacing each unbounded universal (existential) quantifier $\forall x$ ($\exists x$) which occurs in some formula of Γ by $\forall x < m$ ($\exists x < p$). If we think of m, n, p as numerals, $\Gamma_{\vec{n}}[m, p]$ is a set of *bounded sentences*.

We write $\models \Gamma$ (Γ set of sentences) to mean that the disjunction over the sentences of Γ is true under the standard interpretation of number theory.

The relativization A^t of a formula A is obtained by replacing each unbounded quantifier occurring in A by a corresponding quantifier bounded by t (e.g. $\exists x$ becomes $\exists x < t$).

Lemma 1.6.

- *If $n_2 \leq n_1$, $m_1 \leq m_2$ and $\Gamma[n_1, m_1]$, then $\Gamma[n_2, m_2]$;*
- *if A is bounded, $A[n, m] = A$; if A is Σ and B is Π, then*

$$A[n, m] = A^m \text{ and } B[n, m] = B^n$$

If t is a term with free variables in the list $\vec{a}$, δ_t is the primitive recursive function defined by $t(\vec{a})$ in the standard model.

Lemma 1.7 (Majorization)**.** *Let $\mathcal{D} \vdash_1^k \Gamma_{\vec{a}}$. Then we can find a primitive recursive function $f_{\mathcal{D}}$ such that, for every $m > 1$, $n_i \in [0, m)$:*

(i) $\Gamma_{\vec{n}}[m, f_{\mathcal{D}}(m, \vec{n})]$;

(ii) $m \leq f_{\mathcal{D}}(m, \vec{n})$

Proof.
Case 1: $k = 0$. Then $\Gamma_{\vec{a}}$ is an axiom. This case is essentially trivial and we can choose, for arbitrary $m > 1$, and each $n_i \in [0, m)$ in the list $\vec{n}$, $f_{\mathcal{D}}(m, \vec{n}) = m > 1$. Now either $\Gamma_{\vec{n}}[m, f_{\mathcal{D}}(m, \vec{n})] = \Gamma_{\vec{n}}$ and $\Gamma_{\vec{n}}$ is true in the standard model, or else

$$\Gamma_{\vec{n}}[m, f_{\mathcal{D}}(m, \vec{n})] = \Delta_{\vec{n}}, \forall z < m(\neg(z+1) + t_{\vec{n}} = s_{\vec{n}}), t_{\vec{n}} < s_{\vec{n}}.$$

But $\exists z < m((z+1) + t_{\vec{n}} = s_{\vec{n}}) \rightarrow t_{\vec{n}} < s_{\vec{n}}$ holds in the standard model and we are done.

Case 2: $k > 0$.

(Cut) First of all, note that, since the 1-rank of the given derivation is ¡1, then every cut formula occurring in $\mathcal{D}$ is Π_1 or Σ_1. Hence the conclusion of the derivation has the form, for some $k_0, k_1 < k$, and bounded B:

$$\vdash_1^{k_0} \Gamma_{\vec{a}}, \exists x.B \quad \text{and} \vdash_1^{k_1} \Gamma_{\vec{a}}, \forall x.\neg B$$

Then by IH there exist $p_0 = G(m, \vec{n})$, $p_1 = H(m, \vec{n})$, such that

$$m \leq G(m, \vec{n}) \text{ and } m \leq H(m, \vec{n}).$$

and hence, if we choose $F(m, \vec{n}) = H(G(m, \vec{n}), \vec{n})$,

$$\begin{aligned} &m \leq F(m, \vec{n}) \\ &\Gamma_{\vec{n}}[p_0, F(m, \vec{n})], \forall x < p_0.\neg B_{\vec{n}} \\ &\Gamma_{\vec{n}}[m, G(m, \vec{n})], \exists x < p_0.B_{\vec{n}} \end{aligned}$$

By persistence lemma:

$$\begin{aligned} &\Gamma_{\vec{n}}[m, F(m, \vec{n})], \forall x < p_0.\neg B_{\vec{n}} \\ &\Gamma_{\vec{n}}[m, F(m, \vec{n})], \exists x < p_0.B_{\vec{n}} \end{aligned}$$

The conclusion $\Gamma_{\vec{n}}[m, F_{\mathcal{D}}(m, \vec{n})]$ follows by cut.

(∀) Then for some fresh parameter b and $k_0 < k$:

$$\vdash_1^{k_0} \Gamma_{\vec{a}}, A_{\vec{a}}(b)$$

Then by IH there is G_0, such that, for $m > 1$ and $\vec{n}, i \in [0, m)$, $G_0(m, n, i) \geq$ m and

$$\Gamma_{\vec{n}}[m, G_0(m, \vec{n}, i)], A_{\vec{n}}[m, G_0(m, \vec{n}, i)](i).$$

Choose $G(m, \vec{n}) = \sum\{G_0(m, \vec{n}, i) | 0 < i < m\}$: then G is primitive recursive and by persistence:

$$\Gamma_{\vec{n}}[m, G(m, \vec{n})], \forall x < m.A_{\vec{n}}[m, G(m, \vec{n})]$$

(∃)

Then the last inference has the form, for some fresh parameter b and some $k_0 < k$ and some $s_{\vec{a}}$:

$$\vdash_1^{k_0} \Gamma_{\vec{a}}, A_{\vec{a}}(s_{\vec{a}})$$

Then by IH there is G_0, such that, for $m > 1$ and $\vec{n} \in [0, m)$, $G_0(m, \vec{n}) \geq m$ and

$$\Gamma_{\vec{n}}[m, G_0(m, \vec{n})], A_{\vec{n}}[m, G_0(m, \vec{n})](s_{\vec{n}}).$$

Choose $G(m, \vec{n}) = G_0(m, \vec{n}) + \delta_s(\vec{n}) + 1$; then G is primitive recursive and by definition, $G(m, \vec{n}) > \delta_s(\vec{n})$. Hence by persistence:

$$\Gamma_{\vec{n}}[m, G(m, \vec{n})], \exists x < G(m, \vec{n}).A_{\vec{n}}[m, G(m, \vec{n})]$$

($I\Sigma_1$) Then, for some $k_0, k_1 < k$, for some bounded formula B, the final part of the derivation has the form

$$\vdash_1^{k_0} \Gamma_{\vec{a}}, \exists v.B_{\vec{a}}(v, 0)$$
$$\vdash_1^{k_1} \Gamma_{\vec{a}}, \forall x(\exists v.B_{\vec{a}}(v, x) \to \exists v.B_{\vec{a}}(v, x + 1))$$

By inversion, for b fresh parameter:

$$\vdash_1^{k_1} \Gamma_{\vec{a}}, \forall v.\neg B_{\vec{a}}(v, b), \exists v.B_{\vec{a}}(v, b + 1)$$

Then by IH there are functions G, H such that, for every $m > 1$ and $\vec{n}, p \in [0, m)$:

$$m \leq G(m, \vec{n}) \text{ and } m \leq H(m, \vec{n}, p)$$
$$\Gamma_{\vec{n}}[m, G(m, \vec{n})], \exists v < G(m, \vec{n}).B_{\vec{n}}(v, 0)$$
$$\Gamma_{\vec{n}}[m, H(m, \vec{n}, p)], \forall v < m.\neg B_{\vec{n}}(v, p), \exists x < H(m, \vec{n}, p).B_{\vec{n}}(v, p + 1)$$

Define by primitive recursion:

$$\begin{aligned} F(m, \vec{n}, 0) &= G(m, \vec{n}) \\ F(m, \vec{n}, q + 1) &= H(m, \vec{n}, F(m, \vec{n}, q)) \end{aligned}$$

Then check by bounded induction on p, using persistence that $m \leq F(\vec{n}, p)$ and

$$\Gamma_{\vec{n}}[m, F(m, \vec{n}, p)], \exists x < F(m, \vec{n}, p).B_{\vec{n}}(v, p)$$

The cases of the propositional inferences are eft for exercise. □

Let PRA be the system of primitive recursive arithmetic, i.e. $I\Sigma_1$ with the induction rule $I\Sigma_1$ replaced by the same rule restricted to bounded formulas. Then by inspecting the prove of the majorization lemma we see that indeed we have proved:

Theorem 1.8. *If $\Gamma_{\vec{n}}$ is derivable in $I\Sigma_1$, then we can find a term f of PRA such that PRA derives*

$$\forall x \forall \vec{y}.(0 < \vec{y} \leq x \to x \leq f(x, \vec{y}))$$
$$\forall x \forall \vec{y}.(0 < \vec{y} \leq x \to \bigvee \Gamma_{\vec{y}}[x, f(x, \vec{y})])$$

Corollary 1.9 (Parsons-Mints-Takeuti)**.** *If $I\Sigma_1 \vdash \forall x \exists A(x,y)$ (A bounded), than PRA proves $\forall x \exists A(x,y)$.*

Hence the provably recursive functions of $I\Sigma_1$ are exactly the primitive recursive functions.

It also follows that if $I\Sigma_1$ proves an existential statement $\exists x A$, then a bound k can be effectively produced from the proof. However, the *PRA-proofs are in generally hyperexponentially longer than the corresponding proofs that make use of $I\Sigma_1$- induction.*[1] This has been shown by Aleksandar Ignjatovic in his Ph.D. thesis (Berkeley 1990; see also Caldon-Ignjatovic, J. Symbolic Logic Volume 70 (2005), 778-794).

1.3 Refinements for feasible theories

Parikh's theorem Restrict the stock of primitive recursive functions available in the system to an algebra $\mathcal{P}$ of functions which is closed under explicit definitions and contains at least sums and products; assume that all functions in the class are polynomially bounded. Let $I\Delta_0(\mathcal{P})$ be the subtheory of $I\Sigma_1$ which has logical axioms, defining axioms for $\mathcal{P}$ and bounded induction rule (in the new language).

Let Σ-Reflection be the inference $\Sigma - R$: if A is Σ, then

$$\frac{\Gamma, A}{\Gamma, \exists x A^x}$$

The following result is a slight strengthening of a theorem proved by Parikh in 1971.

Theorem 1.10. *If $I\Delta_0(\mathcal{P})$ + Σ-collection proves $\forall x \exists y A(x,y)$ (A bounded), then for some term t, $I\Delta_0(\mathcal{P})$ proves $\forall x \exists y < t.A(x,y)$.*

In particular every provably recursive function of $I\Delta_0(\mathcal{P})$ + Σ-collection is polynomially bounded.

For the proof, apply a (correspondingly modified) majorization argument. It turns out that bounded induction is transformed into itself and the new case of the collection rule is easily disposed of. It follows that the exponential function $x \mapsto 2^x$ is not provably total in the system.

A version of Buss's theorem for bounded arithmetic Another application leads us closer to complexity theoretic issues. We won't give all formal details, but only sketch the mains points. First of all, one replaces the standard arithmetical model with the structure $\mathcal{W}$ of binary words, which includes

(i) the empty sequence, 0, 1, the binary successors $x \mapsto x0$ and $x \mapsto x1$;

(ii) the subword relation $\subset$ among binary words;

[1] More precisely $I\Sigma_1$ *has a non-Kalmar elementary speed-up over* PRA for in the following sense: (i) there is a sequence $\{A_i | i \in \omega\}$ of formulas such that if p_i, q_i are the shortest proofs of A_i in $I\Sigma_1$), PRA (respectively), then *for no elementary function* f, it holds that $|p_i| < |f(q_i)|$:(ii) but *there is a non-elementary function* h such that, for every i, if p_i, q_i are the shortest proofs of A_i in $I\Sigma_1$, PRA (respectively), then $|p_i| < |h(q_i)|$; here $|p|$ stands for the length of the proof p.

(iii) $*$ (word concatenation), $\times$ (word multiplication).

It is essential to observe that binary words can be compared according to two distinct relation:

- the part relation $\subseteq^*$: $a \subseteq^* b$ iff $c * a = b$, for some c;
- the length relation $\leq$: $a \leq b$ iff $1 \times a \subseteq 1 \times b$.

The algebra P of PTIME-functions is then the least collection of functions from binary words to binary words which, besides, constants, projections, subword test (i.e. the characteristic function of $a \subseteq b$) is closed under composition and bounded iteration, i.e. the schema below: if g, h_i, for $i = 0, 1$, t are functions of the appropriate arity in P, then the function defined by

$$\begin{aligned} f(x_1, \ldots, x_n,) &= g(x_1, \ldots, x_n); \\ f(x_1, \ldots, x_n, yi) &= h_i(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y)) \lceil t(x_1, \ldots, x_n) \end{aligned}$$

where in general $a \lceil b$ denotes the truncation of the word a to its initial subword of length $|b|$;[2].

It is useful to keep in mind that

Fact 1.11.

1. $f \in P$ iff f is computable by a Turing machine in time polynomial in the length of the input;
2. the class of P-predicates (i.e. those whose characteristic function is in P) is closed under sharply bounded quantification (i.e. under $\forall x \subseteq^*$, $\exists x \subseteq^*$);
3. a predicate R of binary words is NP iff R is definable in the form

$$R(a) \Leftrightarrow \exists x \leq b. S(x, a),$$

   S being a PTIME;

4. the class of NP-predicates is closed under sharply bounded quantifiers and under $\exists x \leq s$.

In order to define the system PTCA of *polytime computable arithmetic* (in the style of Ferreira[3].), we fix a first order language of $\mathcal{W}$ expanded with function symbols for P-function. Also, inspired by the previous fact, we single out two classes of formulas:

- the class of *sharply bounded formulas* is the least class including atomic formulas, and closed under boolean operations and sharply bounded quantifiers;
- the class of Σ^*-formulas is the least class including atomic formulas, and closed under conjunction, disjunction, sharply bounded quantifiers and unbounded existential quantifiers;

[2] $|b|$ is the number of binary digits of b.

[3] Polynomial time computable arithmetic, in: W. Sieg ed., *Logic and Computation*, Contemporary Mathematics, vol.6 (1990), pp. 137-156

- a formula A is Σ^b_1 if it has the form $\exists x \leq t.B$, with B sharply bounded.

PTCA is the Tait style sequent calculus, which consists of

- its initial sequents which are either standard logical axioms or have the form Γ, E, E being a defining equation of a P-function or a finite set of quantifier free conditions characterizing the basic constructors and relations of the algebra of binary words;
- standard logical rules;
- the rule $PIND$ of induction on notations for sharply bounded formulas A: if $a \notin FV(\Gamma)$,

$$\frac{\Gamma, A(\emptyset) \qquad \Gamma, \forall x(A(x) \rightarrow A(x0)) \qquad \Gamma, \forall x(A(x) \rightarrow A(x1))}{\Gamma, A(a)}$$

NPIND is obtained from PTCA by replacing the $PIND$ with extension of the same rule, but allowing induction Σ^b_1-formulas.

In order to slightly strengthen the Buss-Ferreira theorem, the sharp relativization A^s of A is obtained by replacing each unbounded quantifier Qx ($Q = \forall, \exists$) occurring in A with the corresponding sharply bounded $Qx \subseteq^* s$. We need a modified version of reflection. Let Σ^*-Reflection be the inference $\Sigma^* - R$ which has the same form as $\Sigma - R$, except that A is restricted to be Σ^*.

Theorem 1.12. *Let $A(x, y)$ be Σ^b_1 and $\forall x \exists y A(x, y)$ is derivable in NPIND plus Σ^*-reflection. Then $\forall x \exists y A(x, y)$ is a theorem of PTCA.*

Corollary 1.13. *The provably recursive functions of NPIND plus Σ^*-reflection are exactly those of PTCA and hence the polytime functions.*

The proof of the theorem is constructive and it can be given via a majorization lemma, in analogy with the proof of the PMT-theorem. Using a preliminary Hauptsatz, every cut in the given derivation can be assumed to be Σ^* or Σ^*.

Then the crucial point is (i) to modify the asymmetric interpretation $A[a, b]$, to the extent that every unbounded universal quantifier $\forall x$ ($\exists x$) is replaced by $\forall x \subseteq^* a$ ($\exists x \subseteq^* b$); (ii) to exploit the closure properties of the class of polytime functions.[4]

2 Cut elimination and abstraction

2.1 Cut elimination for a classical predicative system: ACA_0

We show that if contraction is retained and e.g. we live in classical logic, but we assume predicative abstraction, then we do have forms of cut elimination. As a corollary, it is possible to transform every proof of an arithmetical theorem in the system ACA_0 of second order arithmetic with set induction but arithmetical comprehension into an arithmetical proof. The new proof is in general much longer (hyperexponential) speed up.

[4]The majorization arguments of the preceding subsections are contained in A. Cantini, Majorizing provably recursive functions in fragments of PA, *Archiv für Mathematische Logick und Grundlagenforschung*, 25, 1985, 21-31; Asymmetric interpretations for bounded theories, *Mathematical Logic Quarterly*, vol.42 (1996), 270-288

First of all, the language of ACA_0 is a first order language extended with a constant 0, successor, addition $+$, product $\times$, a countable set of set variables X, Y, Z, ... Individual terms are inductively generated from variables, constants and function symbol application. Formulas are inductively generated from atoms $t = s$, $t \in X$ and their negations by means of quantifiers of either sort, $\vee$, $\wedge$.

A is *predicative* if A has no occurrence of bounded set quantifier ($t \in X \to \forall x(x + r = t)$) is predicative, $\exists Y(0 \in Y)$ is not predicative). We inductively assign a notion of *predicative rank* $pr(A)$ to each formula A:

1. $pr(A) = 0$ if A has the form $t = s$, $\neg t = s$, $t \in X$, $\neg t \in X$;
2. $pr(A \wedge B) = pr(A \vee B) = \max(pr(A), pr(B)) + 1$;
3. $pr(\forall x A) = pr(\exists x A) = pr(A) + 1$;
4. $pr(\forall X A) = pr(\exists X A) = \max(pr(A(X)) + 1, \omega)$

We use the abstract $\{x \mid B\}$ as a notation for the predicate defined by B. If $A(X)$ is a formula with X free, B is any formula, $A[X := \{x \mid B\}]$ designates the formula obtained by replacing in $A(X)$ every atom of the form $t \in X$ with $B[x := t]$.[5]

Observe that predicative formulas always have finite predicative rank. Moreover:

Lemma 2.1. *If B is a predicative formula,*

$$pr(A[X := \{x \mid B\}]) < pr(QX.A(X))$$

Definition 2.2.

(2PL) The Tait-style system of *predicative second order logic* 2PL including

- standard logical axioms of the form $\Gamma, \neg A, A$ (A atom);
- standard first order rules and cut;
- the second order quantifier rules: if $V \notin FV(\Gamma)$, B is predicative

$$\frac{\Gamma, A(V)}{\Gamma, \forall X A}\ [\forall_2]; \qquad \frac{\Gamma\,, A[X := \{x \mid B\}]}{\Gamma\,, \exists X A}\ [\exists_2].$$

Below $\mathsf{2PL} \vdash_k \Gamma$ to mean that there exists a derivation $\mathcal{D}$ of Γ in 2PL with $pr(A) < k$, for every cut formula A occurring in $\mathcal{D}$.

(ACA_0) ACA_0 is the subsystem of second order arithmetic based on arithmetical comprehension, consisting of:

- a Hilbert-style axiomatization of two-sorted predicate calculus with equality (with number and set variables);
- standard Peano axioms for 0, successor, plus and times; standard first order rules and cut;

[5] Assuming that renaming of bound variables are carried out in order to avoid clashes of variables

- the arithmetical comprehension schema ACA: if $A(u, \vec{Y})$ is predicative

$$\exists X \forall u(u \in X \leftrightarrow A(u, \vec{Y})) \tag{2.1}$$

- the induction axiom Ind:

$$\forall X((0 \in X \wedge \forall u(u \in X \rightarrow (u+1) \in X)) \rightarrow \forall u(u \in X)) \tag{2.2}$$

(Ax) Let $I_1 = \forall X(\forall x \forall y(x = y \wedge x \in X \rightarrow y \in X)$, $I_0 = Ind$, $I_2 = \forall x(x = x)$, while $I_3, \ldots I_8$ are (in the given order) the sets corresponding to the Peano axioms on successor, plus and times.

Arx is the sequent consisting of the arithmetical axioms $I_2, \ldots, I_8$. Also let $\neg Arx := \{\neg I_2, \ldots, \neg I_8\}$

Note that $\neg I_0$, $\neg I_1$ have the form $\neg \forall X B_0(X)$, $\neg \forall Y B_1(Y), A$, with B_0, B_1 predicative.

Lemma 2.3. *The schema 2.1 is derivable in pure 2PL:*

Using the deduction theorem, we have:

Lemma 2.4. *$ACA_0 \vdash A$ iff 2PL derives the sequent*

$$Arx, \neg \forall X B_0(X), \neg \forall Y B_1(Y), A$$

We then can exploit lemma 2.1 and verify that the cut elimination theorem goes through; in particular:

Theorem 2.5. *If 2PL proves Γ, then there is a cut free proof of the same conclusion in 2PL.*

Proof. (Hint). First of all, check that there is a cut free proof of the tautology lemma $\Gamma, A, \neg A$. Then observe that 2PL-derivability satisfies:

1. Weakening: if $\vdash_k \Gamma$, then $\vdash_k \Gamma, \Delta$;
2. Predicative substitution: if if $\vdash_k \Gamma$, B is predicative, then $\vdash_k \Gamma(V)$ implies then $\vdash_k \Gamma[X := \{x \mid B\}]$ (here the length may increase, but the rank is left unchanged);
3. Inversion:
   - $\vdash_k \Gamma, A \wedge B \Rightarrow \vdash_k \Gamma, A$ and $\vdash_k \Gamma, B$;
   - $\vdash_k \Gamma, A \vee B \Rightarrow \vdash_k \Gamma, A, B$;
   - $\vdash_k \Gamma, \forall x A \Rightarrow \vdash_k \Gamma[a := t]$;
   - $\vdash_k \Gamma, \forall X A \Rightarrow \vdash_k \Gamma[X := \{x \mid B\}]$ (B predicative).

Then one shows a reduction lemma in the form:

(Red_2) If $\vdash_k \Gamma, A$ and $\vdash_k \Delta, \neg A$ with $k \leq pr(A)$, then $\vdash_{pr(A)} \Gamma, \Delta$

The proof is carried out by induction on the sum of the lengths of the given derivations. The case where A is quantified is dealt with by means of the inversion lemma and using lemma 2.1. Iterating the reduction yields the cut free proof. □

Theorem 2.6 (Herbrand for predicative 2^{nd} order logic)**.** *If 2PL proves $\neg\forall X B(X)$ and B is predicative, then we can find predicative formulas $C_1, \ldots, C_k$, such that*

$$\neg B[X := \{x \mid C_1\}], \ldots, \neg B[X := \{x \mid C_k\}]$$

is provable in 2PL (and actually in the first-order fragment which omits second order quantifiers).

Proof. We may assume that there is a cut free derivation $\mathcal{D}$ of $\neg\forall X B(X)$. In $\mathcal{D}$ there are applications of the rule for introducing with the existential quantifier $\neg\forall X B(X)$. Let $\neg B[X := \{x \mid C_1\}]$, $\ldots$, $\neg B[X := \{x \mid C_k\}]$ be the list including exactly the minor formulas of such inferences. Let $\vec{z}$ be the list of all eigenvariables of the inferences of $\mathcal{D}$ introducing $\forall$ on individual variables. Choose

$$\Lambda := \{\exists \vec{z} \neg B[X := \{x \mid C_1\}], \ldots, \exists \vec{z} \neg B[X := \{x \mid C_k\}]\}$$

Λ is a set of predicative formulas. We carry out the following transformation on $\mathcal{D}$:

(a) add Λ to the side formulas of each node in $\mathcal{D}$;

(b) erase each occurrence of $\neg\forall X B(X)$ in $\mathcal{D}$.

It turns out that the resulting derivation $\mathcal{D}^*$ is a derivation of the predicative sequent

$$\neg B[X := \{x \mid C_1\}], \ldots, \neg B[X := \{x \mid C_k\}],$$

provided some contraction and elimination of repetitions. □

Corollary 2.7. *If $ACA_0 \vdash A$ and A is an arithmetical formulas with no second order variable, then $\mathsf{PA} \vdash A$*

For the proof, apply the second order Herbrand theorem and observe that each predicative instance of I_0, I_1 is already provable in PA.

2.2 Cut elimination for a non-classical impredicative system: Grišin's calculus

It was already observed by Fitch in 1936 that the W combinator was responsible for the Russell paradox. We now consider naive comprehension on the ground of the multiplicative additive fragment of affine linear logic (Grišin's logic).

Definition 2.8. Formal language $\mathcal{L}_s$ is the elementary set theoretic language, which comprises

1. the binary predicate symbol $\in$;
2. the logical symbols $\rightarrow$, $\wedge$, $\vee$, $\otimes$, $+$, $\exists$, $\forall$, the propositional constants $\bot$, $\top$.
3. the abstraction operator $\{-|-\}$;
4. individual variables $(x, y, z, \ldots)$.

$\bot$ is the absurd proposition; $\top$ is the true proposition; $\rightarrow$ stands for a substructural implication: $A \rightarrow B$ roughly means that B follows from A via a deduction which uses the assumption A at most once; $\wedge$, $\vee$ ($\otimes$, $+$) denote the so-called additive (multiplicative) conjunction and disjunction of contractionless logic; finally, $\exists$, $\forall$ stand for the usual quantifiers.

The rules The calculus GS is a sequent calculus, which includes the group ID of logical initial sequents, rules for logical operators, quantifiers, $\bot$, and $\in$-introduction rules, which properly deal with set theoretic abstraction. Each inference is assigned a finite ordinal number in square brackets – the $\in$–level –, which is required in the cut elimination procedure (in particular, it is needed to "control" $\in$-rules).

- ID-rule:
$$\Gamma,\, t \in s \;\Rightarrow\; \Delta, t \in s \quad [0]$$
- $\bot$–rule:
$$\Gamma, \bot \;\Rightarrow\; \Delta \quad [0]$$
- $\top$-rule:
$$\Gamma \;\Rightarrow\; \Delta,\, \top \quad [0]$$
- $\rightarrow$-rules:
$$\frac{\Gamma,\, A \;\Rightarrow\; B,\, \Delta \quad [\alpha]}{\Gamma \;\Rightarrow\; A \rightarrow B,\, \Delta \quad [\alpha]} \qquad \frac{\Gamma \;\Rightarrow\; \Delta,\, A \quad [\alpha] \quad \Gamma',\, B \;\Rightarrow\; \Delta' \quad [\beta]}{\Gamma,\, \Gamma',\, A \rightarrow B \;\Rightarrow\; \Delta,\, \Delta' \quad [\alpha+\beta]}$$
- $\forall$-rules :
$$\frac{\Gamma \Rightarrow \Delta, A[x := a] \quad [\alpha]}{\Gamma \;\Rightarrow\; \Delta,\, \forall x A \quad [\alpha]} \qquad \frac{\Gamma, A[x := s] \Rightarrow \Delta \quad [\alpha]}{\Gamma, \forall\, x A \Rightarrow \Delta \quad [\alpha]}$$
  Proviso: $a \notin FV(\Gamma \Rightarrow \Delta, \forall x A)$.
- $\exists$-rules:
$$\frac{\Gamma, A[x := a] \;\Rightarrow \Delta \quad [\alpha]}{\Gamma, \exists x A \;\Rightarrow\; \Delta \quad [\alpha]} \qquad \frac{\Gamma \Rightarrow \Delta, A[x := s] \quad [\alpha]}{\Gamma \Rightarrow \Delta, \exists x A \quad [\alpha]}$$
  Proviso: $a \notin FV(\Gamma,\, \exists x A \Rightarrow \Delta)$.
- $\vee$-rules ($i = 0, 1$):
$$\frac{\Gamma \;\Rightarrow\; \Delta\,, A_i \quad [\alpha]}{\Gamma \;\Rightarrow\; A_1 \vee A_2,\, \Delta \quad [\alpha]} \qquad \frac{\Gamma,\, A \Rightarrow\; \Delta \quad [\alpha] \quad \Gamma,\, B \Rightarrow\; \Delta \quad [\beta]}{\Gamma,\, A \vee B \;\Rightarrow\; \Delta \quad [\alpha+\beta]}$$
- $\wedge$-rules ($i = 0, 1$):
$$\frac{\Gamma \Rightarrow\; \Delta,\, A \quad [\alpha] \quad \Gamma \Rightarrow\; \Delta,\, B \quad [\beta]}{\Gamma \;\Rightarrow\; \Delta, A \wedge B \quad [\alpha+\beta]} \qquad \frac{\Gamma,\, A_i \Rightarrow\; \Delta \quad [\alpha]}{\Gamma,\, A_1 \wedge A_2 \Rightarrow\; \Delta \quad [\alpha]}$$
- $\otimes$-rules:
$$\frac{\Gamma \Rightarrow\; \Delta,\, A \quad [\alpha] \quad \Gamma' \Rightarrow\; \Delta',\, B \quad [\beta]}{\Gamma,\, \Gamma' \;\Rightarrow\; \Delta, \Delta', A \otimes B \quad [\alpha+\beta]} \qquad \frac{\Gamma,\, A\,, B, \Rightarrow\; \Delta \quad [\alpha]}{\Gamma,\, A \otimes B \Rightarrow\; \Delta \quad [\alpha]}$$
- $+$-rules:
$$\frac{\Gamma, A \Rightarrow\; \Delta \quad [\alpha] \quad \Gamma', B \Rightarrow\; \Delta', B \quad [\beta]}{\Gamma,\, \Gamma',\, A + B \Rightarrow\; \Delta, \Delta' \quad [\alpha+\beta]} \qquad \frac{\Gamma \;\Rightarrow\; \Delta,\, A,\, B \quad [\alpha]}{\Gamma \;\Rightarrow\; \Delta,\, A + B \quad [\alpha]}$$

- Cut:
$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad [\alpha] \quad A, \Gamma_2 \Rightarrow \Delta_2 \quad [\beta]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \quad [\alpha + \beta]}$$

- $\in$-rules (in short CA):
$$\frac{\Gamma \Rightarrow \Delta, A[x := t] \quad [\alpha]}{\Gamma \Rightarrow \Delta, t \in \{x|A\} \quad [\alpha + 1]} \qquad \frac{A[x := t], \Gamma \Rightarrow \Delta \quad [\alpha]}{t \in \{x|A\}, \Gamma \Rightarrow \Delta \quad [\alpha + 1]}$$

Hauptsatz

Theorem 2.9 (Grišin)**.** GS enjoys cut elimination.

Corollary 2.10. GS is consistent

In order to prove the theorem, it is enough to verify

Lemma 2.11. Assume $\mathcal{D}_0 \vdash \Gamma_1 \Rightarrow \Delta_1, A$ and $\mathcal{D}_1 \vdash A, \Gamma_2 \Rightarrow \Delta_2$, where

- the $\in$-level of $\mathcal{D}_0$ is α and the $\in$-level of $\mathcal{D}_1$ is β;
- $\mathcal{D}_0$, $\mathcal{D}_1$ are cut–free.

Then there exists a cut free derivation $\mathcal{D}^*$ of $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ with $\in$-level $\leq \alpha + \beta$.

Proof. Hint: argue by main induction on $\alpha + \beta$ and two subordinate inductions on the logical complexity $|A|$ of the cut formula, and on the sum of the heights of $\mathcal{D}_0$ and $\mathcal{D}_1$. $|A|$ simply counts the number of logical operations above its atomic subformulas. The strategy is standard and there is no special difficulty; one needs induction on the sum of the $\in$-levels, in order to handle the case where the cut formula A is active on both sides and the final rules are exactly $\in$-rules. Indeed, assume we have two derivations, which both conclude with $\in$-rules:
$$\frac{\Gamma_1 \Rightarrow \Delta_1, t \in \{x \mid A\} \quad [i+1] \quad t \in \{x \mid A\}, \Gamma_2 \Rightarrow \Delta_2 \quad [k+1]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \quad [i+k+2]}$$
Then we can obtain the same conclusion if we apply Cut to the immediate subderivations:
$$\frac{\Gamma_1 \Rightarrow \Delta_1, A[x := t] \ [i]; \quad \Gamma_2, A[x := t] \Rightarrow \Delta_2 \ [k]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \quad [i + k]}$$
Even if the cut complexity is increased, the $\in$-level is decreased, we can apply the main induction hypothesis and we are done.
Case $[\in, \rightarrow]$: the final inferences have the form:
$$\frac{\Gamma \Rightarrow \Delta, A[x := t], B \rightarrow C \quad [i]}{\Gamma \Rightarrow \Delta, t \in \{x \mid A\}, B \rightarrow C \quad [i+1]} \qquad \frac{\Gamma_1, C \Rightarrow \Delta_1 \ [k_0]; \quad \Gamma_2 \Rightarrow \Delta_2, B \ [k_1]}{\Gamma_1, \Gamma_2, B \rightarrow C \Rightarrow \Delta_1, \Delta_2 \quad [k_0 + k_1]}$$
Let $\Gamma' := \Gamma_1, \Gamma_2$ and $\Delta' := \Delta_1, \Delta_2$. Then
$$\frac{\Gamma \Rightarrow \Delta, A[x := t], B \rightarrow C \ [i]; \quad \Gamma', B \rightarrow C \Rightarrow \Delta' \ [k_0 + k_1]}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A[x := t] \quad [i + k_0 + k_1]}$$
Since $i + k_0 + k_1 < (i+1) + k_0 + k_1$, we have by main induction hypothesis and $\in$-introduction a cut-free derivation of
$$\Gamma, \Gamma' \Rightarrow \Delta, \Delta', t \in \{x \mid A\} \quad [i + k_0 + k_1 + 1].$$
NB: in this case we could not have used a main induction on $max\{i, k\}$. □

Remark 2.12. According to the tradition (Russell and Poincaré), unrestricted abstraction is classified as *impredicative*. This is also reflected by the fact that the size of the premiss in the introduction rules for abstraction does not usually decrease. This is also seen as (at a first sight) an obstacle to the Hauptsatz. But the theorem above shows that this holds only if impredicative abstraction lives in a logical world, where contraction is admissible. If contraction is absent, the Hauptsatz may keep an entirely elementary and constructive verification.

Digression: undecidability. The calculus GS is not only consistent, but is *computationally complete*. This can be seen in a number of steps. The basic move is to prove that a fixed point theorem holds true without use of standard logic: contraction free logic is enough.

Indeed, although the underlying logic of GS is decidable,[6] GS is not. Since the undecidability argument essentially uses the availability of cut free derivations, we like to include a sketch of the argument.[7]

We define equality (à la Leibniz), extensional equality and (intensional) versions of singleton, intersection and unordered pairing.

Definition 2.13.

$$\begin{array}{rcl} t = s & \Leftrightarrow & \forall z(t \in z \rightarrow s \in z); \\ t =_e s & \Leftrightarrow & \forall z(z \in t \rightarrow z \in s) \otimes \forall z(z \in s \rightarrow z \in t); \\ \emptyset & := & \{x|\bot\}; \\ \{t\} & := & \{x|x = t\}; \\ t \cap s & := & \{x|x \in t \otimes x \in s\}; \\ \{t, s\} & := & \{x|x = t \vee x = s\}. \end{array}$$

The extensionality principle Ext is expressed in sequent form:

$$\Gamma, t =_e s, t \in r \Rightarrow s \in r.$$

Lemma 2.14 (provable in GS)**.**

$$\begin{array}{rcl} & \Rightarrow & t = t; \\ t = s, A[x := t] & \Rightarrow & A[x := s]; \\ t = s & \Rightarrow & s = t; \\ t = s, s = r & \Rightarrow & t = r \\ t = s & \Rightarrow & t = s \otimes t = s; \\ t = s & \Rightarrow & t =_e s. \end{array}$$

The verification is easily carried out in the system GS.

Let GS_r be the subsystem which only contains those instances of $\in$-rules that grant the existence of $\{a\}$, $a \cap b$.

Lemma 2.15 ($\in$-Contraction)**.** GS_r+ Ext *proves the sequent*

$$t \in s \Rightarrow t \in s \otimes t \in s.$$

[6]Historically, the observation that predicate calculus without contraction is decidable is already present in Hao Wang 1963. For more details, see Ono's paper in the reference list.

[7]For more details, see A. Cantini, The undecidability of Grišin's set theory, *Studia Logica*, vol.74, 345-368, 2003.

Proposition 2.16 (Grišin's paradox)**.** GS+ Ext *is inconsistent.*

Proof. By $\in$-rules for arbitrary formulas, we have

$$A \Rightarrow A \otimes A.$$

Hence the system with extensionality comprises standard positive logic and type free comprehension, and it is inconsistent by Curry's paradox. □

Rather surprisingly, if we define $\langle a, b\rangle := \{\{a\}, \{a, b\}\}$, we obtain:

Proposition 2.17 (Ordered pairing; provable in GS)**.**

$$\langle a, b\rangle = \langle c, d\rangle \Rightarrow a = c \otimes b = d.$$

Theorem 2.18. *If* $fa := \{x \mid \langle x, a\rangle \in f\}$*, then there is a term* I_f *with* $FV(I_f) = f$ *such that, provably in* GS*:*

$$\Rightarrow I_f =_e fI_f$$

Proof. Hint (complete by exercise). Choose

$$D_f = \{z|\exists x \exists g(z = \langle x, g\rangle \otimes x \in f(gg))\} \tag{2.3}$$

$$I_f = D_f D_f \tag{2.4}$$

Then apply comprehension, the $=$-contraction lemma, ordered pairing and the cut rule. □

Representing combinatory logic CL Recall that the relation "$t = s$ is equationally provable in combinatory logic", i.e. formally $\mathsf{CL} \vdash t = s$ is the smallest equivalence relation on terms, generated by the initial conditions $Kab = a$ and $Sabc = ac(bc)$, and closed under the inferences:

$$\begin{aligned} a = b &\Rightarrow ac = bc \\ a = b &\Rightarrow ca = cb \end{aligned}$$

Theorem 2.19. *CL* is essentially undecidable.

Let TER_{CL} be the set of CL-terms and let TER_{GS} be the set of GS-terms. Then:

Theorem 2.20. *There exist:*

(i) a translation $\widehat{t} : TER_{CL} \mapsto TER_{GS}$

(ii) a closed term $\mathcal{E}$ *in* GS *such that*

$$\mathsf{CL} \vdash t = s \Leftrightarrow \mathsf{GS} \vdash \Rightarrow \langle \widehat{t}, \widehat{s}\rangle \in \mathcal{E}$$

Hence GS *is undecidable*

As to the main steps the proof, by fixed point we can simulate the syntax of CL, the definition of CL-derivability and natural numbers.

For instance, if we define

$$\begin{array}{rcl} \overline{0} & := & \emptyset; \\ t+1 & := & \{t\}; \\ \overline{n+1} & := & \overline{n}+1, \end{array}$$

it is straightforward to check that the successor axioms become provable and there exists a closed term ω representing the set of natural numbers.

By application of the contraction free nature of the calculus (e.g., restricted invertibility of the $\exists$-introduction rule to the right, given that the antecedent is empty), it is not difficult to check:

Lemma 2.21. *1. if* $\mathsf{GS} \vdash \Rightarrow t = s$ *, then* $t \equiv s$ *("the literal identity property");*

2. if $\mathsf{GS} \vdash \Rightarrow t \in \omega$*, then for some natural number* n*,* $t \equiv \overline{n}$ *("the* ω*–evaluation property").*

Appendix

(A) The propositional logic underlying GS is also known as IML, the logic of involutive ML-algebras.

is a commutative integral bounded residuated lattice, i.e. a structure

$$\langle L, \vee, \wedge, \otimes, \rightarrow, \top, \bot \rangle$$

such that

1. $\langle L, \vee, \wedge \rangle$ is a lattice with maximum $\top$, minimum $\bot$;
2. $\langle L, \otimes, \top \rangle$ is a commutative semigroup with unit $\top$;
3. $\otimes$ and $\rightarrow$ form an adjoint pair: for all $x, y, z \in L$, $x \leq (y \rightarrow z)$ iff $x \otimes y \leq z$.

Define:

$$\neg x = (x \rightarrow \top); \quad x + y = \neg(\neg x \otimes \neg y)$$

An ML-algebras is **involutive** if it satisfies:

1. INV: $\neg\neg x = x$;

NB: adding contraction $x \otimes x = x$ to an ML-algebra has the effect of collapsing it into a boolean algebra.

(B) The calculus GS can be conservatively to a system GS+K4 extended with a K4-modality.

Lemma 2.22. *The following inferences are admissible for* GS+K4*:*

$$\frac{\Box A, \Box A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta, B}$$

Lemma 2.23 (Löb's rule)**.** *If* $\mathsf{GS4}+\mathsf{K4} \vdash \Box A \Rightarrow A$*, then* $\mathsf{GS4} \vdash \Rightarrow A$*.*

Proof. Exercise: apply Curry's paradox... □

Theorem 2.24 (Löb's principle)**.** $\mathsf{GS4}+\mathsf{K4} \vdash \Rightarrow \Box(\Box A \to A) \to \Box A$.

It follows that GS cannot be endowed with the exponentials of linear logic; this fact leads to light versions of linear logic in investigating implicit computational complexity.

3 Lambda calculi with types: a short introduction

The lectures of this part have closely followed sections 3-4 of Barendregt [2] and hence we do not include the notes.

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