Synthetic Computability
(Computability Theory without Computers)

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What is “synthetic” mathematics?

- Suppose we want to study mathematical structures forming a category $\mathcal{C}$, such as:
  - smooth manifolds and differentiable maps
  - topological spaces and continuous maps
  - computable sets and computable maps
- **Classical approach:** structures are *sets equipped with extra structure*, morphisms *preserve the structure*.
- **Synthetic approach:** embed $\mathcal{C}$ in a suitable *mathematical universe* $\mathcal{E}$ (a model of intuitionistic set theory) and view structures as *ordinary sets* and morphisms as *ordinary maps* inside $\mathcal{E}$.
- The synthetic approach is unusual because:
  (a) construction of a suitable $\mathcal{E}$ may be technically involved,
  (b) unusual axioms and logic may be valid in $\mathcal{E}$.
- But we can forget (a) once it has been done.
- As logicians, we are delighted about (b) because it takes us to exciting new worlds of mathematics.

A synthetic universe for computability theory

- M. Hyland's *effective topos* Eff is the mathematical universe suitable for computability theory.
- In Eff all objects and morphisms are equipped with computability structure.
- We need not know how Eff is built—we just use the logic and axioms which are valid in it.
- Our task is to see *what mathematics is like for those who live inside* Eff.
- Other synthetic worlds:
  - Synthetic Differential Geometry [Lawvere & Kock]
  - Synthetic Topology [Taylor, Escardó]

External and internal view

Comparison of concepts as viewed by us (externally) and by mathematicians inside Eff (internally):

Symbol	External	Internal
$\mathbb{N}$	natural numbers	natural numbers
$\mathbb{R}$	*computable* reals	*all* reals
$f : \mathbb{N} \to \mathbb{N}$	*computable* map	*any* map
$e : \mathbb{N} \twoheadrightarrow A$	*computable* enumeration of A	*any* enumeration of A
$\{\mathrm{true}, \mathrm{false}\}$	truth values	decidable truth values
Ω	truth values of Eff	truth values
$\forall x$	for all *computable* x	for all x
$\exists x$	there exists *computable* x	there exists x
$P \vee \neg P$	decision procedure for P	P or not P

Translating between external and internal views

- There is a systematic way to translate any internal statement to an external one, namely the *interpretation of intuitionistic logic and set theory* in Eff.
- But we will not do that. Instead we shall speak and think like mathematicians inside Eff.

Related Work

- Friedman [1971], axiomatizes coding and universal functions
- Moschovakis [1971] & Fenstad [1974], axiomatize computations and subcomputations
- Hyland [1982], effective topos
- Richman [1984], an axiom for effective enumerability of partial functions, extended in Bridges & Richman [1987]
- We shall follow Richman [1984] in style, and borrow ideas from Rosolini [1986], Berger [1983], and Spreen [1998].

Outline

Introduction

Constructive Mathematics

Basic Computability
- Theorems for Free
- Axiom of Enumerability

Further Results
- The topological view
- Markov Principle
- Recursion Theorem

Appendix: Recursive Analysis
- Inseparable Sets
- The Kleene Tree
- Real Analysis

Conclusion

Intuitionistic logic

- Law of Excluded Middle:

$$\forall p \in \Omega \, . \, (p \vee \neg p)$$

"For every proposition p, p or not p."

- Accepted in classical mathematics (99.9% of mathematicians and falling).
- In intuitionistic mathematics it can only be used in special cases, when p is *decidable*.

Axiom of Choice

- Axiom of Choice:

  For every family $\{A_i\}_{i \in I}$ of non-empty sets there is a choice function $f : I \to \bigcup_{i \in I} A_i$, such that $f(i) \in A_i$ for all $i \in I$.

- Logical formulation of Axiom of Choice: every total relation has a choice function,

$$(\forall x \in A \,.\, \exists y \in B \,.\, R(x, y)) \implies \exists f \in B^A \,.\, \forall x \in A \,.\, R(x, f(x)).$$

- Axiom of Choice $\implies$ Excluded Middle.
- Constructive mathematics: only *Countable Choice*, which is the case $I = A = \mathbb{N}$.

Basic sets and constructions

- Basic sets:

$$\emptyset, \quad \mathsf{1} = \{*\}, \quad \mathbb{N} = \{0, 1, 2, \ldots\}$$

- Set operations:

$$A \times B, \quad A + B, \quad B^A = A \to B, \quad \{x \in A \mid p(x)\}, \quad \mathcal{P}A$$

- We say that A is
  - *non-empty* if $\neg \forall\, x \in A \,.\, \bot$,
  - *inhabited* if $\exists\, x \in A \,.\, \top$.

Some interesting sets

- The set of truth values:

$$\Omega = \mathcal{P}\mathsf{1}$$
$$\text{truth } \top = \mathsf{1}, \quad \text{falsehood } \bot = \emptyset$$

- The set of *decidable* truth values:

$$\mathsf{2} = \{0, 1\} = \{p \in \Omega \mid p \vee \neg p\} ,$$

  where we write $1 = \top$ and $0 = \bot$.
- The set of *classical* truth values:

$$\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p = p\} .$$

- $\mathsf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$.

Decidable and classical sets

- A subset $S \subseteq A$ is equivalently given by its characteristic map
$$\chi_S : A \to \Omega, \qquad \chi_S(x) = (x \in S).$$
- A subset $S \subseteq A$ is *decidable* if $\chi_S : A \to \mathsf{2}$, equivalently
$$\forall x \in A . (x \in S \lor x \notin S) \ .$$
- A subset $S \subseteq A$ is *classical* if $\chi_S : A \to \Omega_{\neg\neg}$, equivalently
$$\forall x \in A . (\neg(x \notin S) \implies x \in S) \ .$$

Enumerable & finite sets

- A is *finite* if there exist $n \in \mathbb{N}$ and a surjection

$$e : \{1, \ldots, n\} \twoheadrightarrow A,$$

  called a *listing* of A. An element may be listed more than once.
- A is *enumerable (countable)* if there exists a surjection

$$e : \mathbb{N} \twoheadrightarrow \mathsf{1} + A,$$

  called an *enumeration* of A. For inhabited A we may take $e : \mathbb{N} \twoheadrightarrow A$.
- A is *infinite* if there exists an injective $a : \mathbb{N} \rightarrowtail A$.

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Lawvere $\to$ Cantor

Theorem (Lawvere)

If $e : A \twoheadrightarrow B^A$ is surjective then B has the fixed point property: for every $f : B \to B$ there is $x_0 \in B$ such that $f(x_0) = x_0$.

Proof.

Given $f : B \to B$, define $g(y) = f(e(y)(y))$. Because e is surjective there is $x \in A$ such that $e(x) = g$. Then $e(x)(x) = f(e(x)(x))$, so $x_0 = e(x)(x)$ is a fixed point of f. □

Corollary (Cantor)

There is no surjection $e : A \twoheadrightarrow \mathcal{P}A$.

Proof.

$\mathcal{P}A = \Omega^A$ and negation $\neg : \Omega \to \Omega$ does not have a fixed point. □

Non-enumerability of Cantor and Baire space

Are there any sets which are *not* enumerable?
Yes, for example $\mathcal{P}\mathbb{N}$, and also:

Corollary

$2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are not enumerable.*

Proof.

2 and $\mathbb{N}$ do not have the fixed-point property. □

We have proved our first synthetic theorem:

Theorem (external translation of above corollary)

The set of recursive sets and the set of total recursive functions cannot be computably enumerated.

Projection Theorem

Recall: the *projection* of $S \subseteq A \times B$ is the set

$$\{x \in A \mid \exists y \in B \,.\, \langle x, y \rangle \in S\} \;.$$

Theorem (Projection)

A subset of $\mathbb{N}$ *is enumerable iff it is the projection of a decidable subset of* $\mathbb{N} \times \mathbb{N}$.

Proof.

If A is enumerated by $e : \mathbb{N} \to 1 + A$ then A is the projection of the *graph* of e, which is the set $\{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \mid m = e(n)\}$.
If A is the projection of $B \subseteq \mathbb{N} \times \mathbb{N}$, define $e : \mathbb{N} \times \mathbb{N} \to 1 + A$ by

$$e\langle m, n \rangle = \texttt{if } \langle m, n \rangle \in B \texttt{ then } m \texttt{ else } \star \;.$$

□

Semidecidable sets

- A *semidecidable truth value* $p \in \Omega$ is one that is equivalent to

$$\exists n \in \mathbb{N} \,.\, d(n)$$

  for some $d : \mathbb{N} \to \mathsf{2}$.

- The set of semidecidable truth values:

$$\Sigma = \{p \in \Omega \mid \exists d \in \mathsf{2}^{\mathbb{N}} \,.\, (p \iff \exists n \in \mathbb{N} \,.\, d(n))\} \,.$$

  This is Rosolini's *dominance*.

- $\mathsf{2} \subseteq \Sigma \subseteq \Omega$.
- A subset $S \subseteq \mathbb{N}$ is *semidecidable* if $\chi_S : A \to \Sigma$.

Semidecidable subsets of $\mathbb{N}$

Theorem

The enumerable subsets of $\mathbb{N}$ are the semidecidable subsets of $\mathbb{N}$.

Proof.

By Projection Theorem, an enumerable $A \subseteq \mathbb{N}$ is the projection of a decidable $B \subseteq \mathbb{N} \times \mathbb{N}$. Then $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, \langle n, m \rangle \in B$.
Conversely, if $A \in \Sigma^{\mathbb{N}}$, by Number Choice there is $d : \mathbb{N} \times \mathbb{N} \to 2$ such that $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, d(m, n)$. □

The enumerable subsets of $\mathbb{N}$:

$$\mathcal{E} = \Sigma^{\mathbb{N}} \,.$$

Partial functions

- A partial function $f : A \rightharpoonup B$ is a function $f : A' \to B$ defined on a subset $A' \subseteq A$, called the *domain* of f.
- Write $f(x)\downarrow$ when $x \in A'$ and $f(x)$ is defined.
- We can view $f : A \rightharpoonup B$ as an ordinary function $f' : A \to \widetilde{B}$ where

$$\begin{aligned} \widetilde{B} &= \{s \in \mathcal{P}B \mid \text{every two elements of } s \text{ are equal}\}, \\ f'(x) &= \{y \in B \mid f(x)\downarrow \wedge y = f(x)\}. \end{aligned}$$

Then

$$\begin{aligned} f'(x) = \{y\} &\iff f(x)\downarrow \wedge f(x) = y, \\ f'(x) = \emptyset &\iff f(x) \text{ undefined.} \end{aligned}$$

Σ-partial functions

The *graph* of $f : \mathbb{N} \rightharpoonup \mathbb{N}$ is the set

$$\{\langle x, y\rangle \in A \times B \mid f(x)\downarrow \wedge y = f(x)\}.$$

When does a partial function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ have an enumerable graph?

Proposition

$f : \mathbb{N} \to \widetilde{\mathbb{N}}$ has an enumerable graph iff $f(n)\downarrow \in \Sigma$ for all $n \in \mathbb{N}$.

Define the *lifting* operation

$$A_\perp = \{s \in \widetilde{A} \mid (\exists\, x \in A \,.\, x \in s) \in \Sigma\}\ .$$

For $f : A \to B$ define $f_\perp : A_\perp \to B_\perp$ to be

$$f_\perp(s) = \{f(x) \mid x \in s\}\ .$$

A *Σ-partial function* is a function $f : A \to B_\perp$.

Domains of Σ-partial functions

Proposition

A subset is semidecidable iff it is the domain of a Σ-partial function.

Proof.

A semidecidable subset $S \in \Sigma^A$ is the domain of its characteristic map $\chi_S : A \to \Sigma = 1_\perp$.
If $f : A \to B_\perp$ is Σ-partial then its domain is the set $\{x \in A \mid f(x)\downarrow\}$, which is obviously semidecidable. □

Theorem (External translation)

A set is semidecidable iff it is the domain of a partial recursive map.

The Single-Value Theorem

A *selection* for $R \subseteq A \times B$ is a partial map $f : A \rightharpoonup B$ such that, for every $x \in A$,

$$(\exists y \in B \,.\, R(x, y)) \implies f(x)\downarrow \wedge R(x, f(x)) \,.$$

This is like a choice function, expect it only chooses when there is something to choose from.

Theorem (Single Value Theorem)

Every semidecidable relation $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$ has a Σ-partial selection.

Axiom of Enumerability

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

Let $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be an enumeration.

Proposition

Σ and $\mathcal{E}$ have the fixed-point property.

Proof.

By Lawvere, $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E} = \Sigma^{\mathbb{N}} \cong \Sigma^{\mathbb{N}\times\mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

The Law of Excluded Middle Fails

The Law of Excluded Middle says $\mathsf{2} = \Omega$.

Corollary

The Law of Excluded Middle is false.

Proof.

Among the sets $\mathsf{2} \subseteq \Sigma \subseteq \Omega$ only the middle one has the fixed-point property, so $\mathsf{2} \neq \Sigma \neq \Omega$. □

Enumerability of $\mathbb{N} \to \mathbb{N}_\perp$

Proposition

$\mathbb{N} \to \mathbb{N}_\perp$ is enumerable.

Proof.

Let $V : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$ be an enumeration. By Single-Value Theorem and Number Choice, there is $\varphi : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\perp)$ such that φ_n is a selection of V_n. The map φ is surjective, as every $f : \mathbb{N} \to \mathbb{N}_\perp$ is the only selection of its graph. □

Corollary (Formal Church's Thesis)

$\mathbb{N}^\mathbb{N}$ is subenumerable (because $\mathbb{N}^\mathbb{N} \subseteq \mathbb{N}_\perp^\mathbb{N}$).

In other words, $\forall f \in \mathbb{N}^\mathbb{N} \,.\, \exists n \in \mathbb{N} \,.\, f = \varphi_n$.

End of Part I

Walk around and rest your brain for 10 minutes.

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The Topological View

- The topological view:

$$\text{semidecidable subsets} = \text{open subsets} .$$

- Σ is the *Sierpinski space*: the space on two points $\bot, \top$ with $\{\top\}$ open and $\{\bot\}$ closed.
- The *topology of A* is Σ^A.

Continuous maps

Theorem

All functions are continuous.

Proof.

Given any $f : A \to B$ and $U \in \Sigma^B$, the set $f^{-1}(U)$ is open because it is classified by $U \circ f : A \to \Sigma$. $\square$

Topological Exterior and Creative Sets

- The *exterior* of an open set is the largest open set disjoint from it.
- An open set $U \in \Sigma^A$ is *creative* if it is without exterior: every $V \in \Sigma^A$ disjoint from U can be enlarged and still be disjoint from U.

Theorem

There exists a creative subset of $\mathbb{N}$.

Proof.

The familiar $K = \{n \in \mathbb{N} \mid n \in \mathsf{W}_n\}$ is creative. Given any $V \in \mathcal{E}$ with $V = \mathsf{W}_k$ and $K \cap V = \emptyset$, we have $k \notin V$ and $k \notin K$, so $V' = V \cup \{k\}$ is larger and still disjoint from K. □

Immune and Simple Sets

- A set is *immune* if it is neither finite nor infinite.
- A set is *simple* if it is open and its complement is immune.

Theorem

There exists a (closed) immune subset of $\mathbb{N}$.

Proof.

Following Post, consider $P = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \wedge n \in W_m\}$, and let $f : \mathbb{N} \to \mathbb{N}_\perp$ be a selection for P. Then $S = \{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N} \,.\, f(m) = n\}$ is the complement of the set we are looking for.
Because $f(m) > 2m$ the set $\mathbb{N} \setminus S$ cannot be finite.
For any infinite enumerable set $U \subseteq \mathbb{N} \setminus S$ with $U = \mathsf{W}_m$, we have $f(m)\downarrow$, $f(m) \in \mathsf{W}_m = U$, and $f(m) \in S$, hence U is not contained in $\mathbb{N} \setminus S$. □

Markov Principle

- If a binary sequence $a \in \mathbf{2}^{\mathbb{N}}$ is not constantly 0, does it contain a 1?
- For $p \in \Sigma$, does $p \neq \bot$ imply $p = \top$?
- Is $\Sigma \subseteq \Omega_{\neg\neg}$?

Axiom (Markov Principle)

A binary sequence which is not constantly 0 *contains a* 1.

Post's Theorem

Theorem

For all $p \in \Omega$,

$$p \in \mathsf{2} \iff p \in \Sigma \wedge \neg p \in \Sigma \; .$$

Proof.

$\Rightarrow$ If $p \in \mathsf{2}$ then $\neg p \in \mathsf{2}$, therefore $p, \neg p \in \mathsf{2} \subseteq \Sigma$.

$\Leftarrow$ If $p \in \Sigma$ and $\neg p \in \Sigma$ then $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore

$$p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top \; ,$$

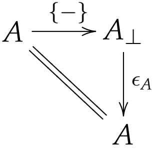
as required.

□

Go out,
count the ducks,
come back.

Focal sets

- A *focal set* is a set A together with a map $\epsilon_A : A_\perp \to A$ such that $\epsilon_A(\{x\}) = x$ for all $x \in A$:



  The *focus* of A is $\perp_A = \epsilon_A(\perp)$.
- A lifted set $A_\perp$ is always focal (because lifting is a monad whose unit is $\{-\}$).

Enumerable focal sets

- Enumerable focal sets, known as *Eršov complete sets*, have good properties.
- A *flat domain* $A_\perp$ is focal. It is enumerable if A is decidable and enumerable.
- If A is enumerable and focal then so is $A^{\mathbb{N}}$:

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{e_\perp^{\mathbb{N}}} A_\perp^{\mathbb{N}} \xrightarrow{\epsilon_A^{\mathbb{N}}} A^{\mathbb{N}}$$

- Some enumerable focal sets are

$$\Sigma^{\mathbb{N}}, \quad \mathsf{2}_\perp^{\mathbb{N}}, \quad \mathbb{N}_\perp^{\mathbb{N}} \; .$$

Multi-valued functions

- A *multi-valued function* $f : A \rightrightarrows B$ is a function $f : A \to \mathcal{P}B$ such that $f(x)$ is inhabited for all $x \in A$.
- This is equivalent to having a *total relation* $R \subseteq A \times B$. The connection between f and R is

$$f(x) = \{y \in B \mid R(x, y)\}$$
$$R(x, y) \iff y \in f(x)\ .$$

- A *fixed point* of $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Recursion Theorem

Theorem (Recursion Theorem)

Every $f : A \rightrightarrows A$ on enumerable focal A has a fixed point.

Proof.

Let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration, and $\epsilon : A_\perp \to A$ a focal map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $e(m) \in f(e(k))$. By Number Choice there is a map $c : \mathbb{N} \to \mathbb{N}$ such that $e(c(k)) \in f(e(k))$ for every $k \in \mathbb{N}$. It suffices to find k such that $e(c(k)) = e(k)$ since then $x = e(k)$ is a fixed point for f.
For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\epsilon(e_\perp(c_\perp(\varphi_m(m)))) = e(n)$. By Number Choice there is $g : \mathbb{N} \to \mathbb{N}$ such that $\epsilon(e_\perp(c_\perp(\varphi_m(m)))) = e(g(m))$ for every $m \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $g = \varphi_j$. Let $k = g(j)$. Then

$$e(k) = e(g(j)) = \epsilon(e_\perp(c_\perp(\varphi_j(j)))) == e(c(g(j))) = e(c(k)) \ .$$

□

Can someone make this proof more beautiful?

Classical Recursion Theorem

Corollary (Classical Recursion Theorem)

For every $f : \mathbb{N} \to \mathbb{N}$ *there is* $n \in \mathbb{N}$ *such that* $\varphi_{f(n)} = \varphi_n$.

Proof.

In Recursion Theorem, take the enumerable focal set $A = \mathbb{N}_\perp^\mathbb{N}$ and the multi-valued function

$$F(g) = \{h \in \mathbb{N}_\perp^\mathbb{N} \mid \exists\, n \in \mathbb{N} \,.\, g = \varphi_n \wedge h = \varphi_{f(n)}\} \ .$$

There is g such that $g \in F(g)$. Thus there exists $n \in \mathbb{N}$ such that $\varphi_n = g = h = \varphi_{f(n)}$. □

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End of Part III

Please vote:

1. End now, please!
2. Go on until 7:30 without a pause.

Plotkin's Domain $\mathbf{2}_{\perp}^{\mathbb{N}}$

- In a *partially ordered set* $(P, \leq$ we say that x and y are *incomparable* if $x \not\leq y$ and $y \not\leq x$.
- The set of Σ-partial binary functions $\mathbf{2}_{\perp}^{\mathbb{N}}$ is a partially ordered:

$$f \leq g \iff \forall n \in \mathbb{N} \,.\, (f(n)\downarrow \implies g(n)\downarrow \wedge f(n) = g(n))$$

  It is called *Plotkin's universal domain* in domain theory.
- Suppose P is a partially ordered set and $x \in P$. Is there a maximal element of P above x?

Inseparable sets

Theorem

There exists an element of $\mathbf{2}_\perp^{\mathbb{N}}$ that is inconsistent with every maximal element of $\mathbf{2}_\perp^{\mathbb{N}}$.

Proof.

Because $\mathbf{2}_\perp$ is focal and enumerable, $\mathbf{2}_\perp^{\mathbb{N}}$ is as well. Let $\psi : \mathbb{N} \twoheadrightarrow \mathbf{2}_\perp^{\mathbb{N}}$ be an enumeration, and let $t : \mathbf{2}_\perp \to \mathbf{2}_\perp$ be the isomorphism $t(x) = \neg_\perp x$ which exchanges 0 and 1, and fixes $\perp$. Consider $a \in \mathbf{2}_\perp^{\mathbb{N}}$ defined by $a(n) = t(\psi_n(n))$. If $b \in \mathbf{2}_\perp^{\mathbb{N}}$ is maximal with $b = \psi_k$, then $a(k) = \neg\psi_k(k) = \neg b(k)$. Because $a(k)$ and $b(k)$ are both total and different they are inconsistent. Hence a and b are inconsistent. □

Binary Trees

- Let $\mathsf{2}^*$ be the set of finite binary sequences, with prefix-ordering $\preceq$.
- The *length* of $[a_0, \ldots, a_{n-1}] \in \mathsf{2}^*$ is $|a| = n$.
- A *tree* $T \subseteq \mathsf{2}^*$ is an inhabited prefix-closed subset.
- A *Kleene tree* T_K is a tree such that:
  1. T_K is decidable (as a subset of $\mathsf{2}^*$),
  2. T_K is unbounded: $\forall k \in \mathbb{N} \,.\, \exists a \in T_K \,.\, |a| \geq k$,
  3. every infinite path exits T_K:

$$\forall \alpha \in \mathsf{2}^{\mathbb{N}} \,.\, \exists n \in \mathbb{N} \,.\, [\alpha_0, \ldots, \alpha_n] \notin T_K \,.$$

Construction of a Kleene Tree

1. Recall an enumeration $\psi : \mathbb{N} \twoheadrightarrow \mathsf{2}^{\mathbb{N}}_{\perp}$ and $s(n) = \neg_{\perp}\psi_n(n)$ which is inconsistent with every $\alpha \in \mathsf{2}^{\mathbb{N}}$.
2. Let $\langle m_{-}, d_{-}\rangle : \mathbb{N} \to \mathbb{N} \times \mathsf{2}$ be an enumeration of the graph of s, i.e., $s(m_k) = d_k$ for all $k \in \mathbb{N}$ and we enumerate all such pairs.
3. Given $a = [a_0, \ldots, a_n] \in \mathsf{2}^*$, say that *a clashes with* $\langle m_{-}, d_{-}\rangle$, if there is $k \leq n$ such that $m_k \leq n$ and $a_{m_k} \neq d_k$.
4. Define $K_T = \{a \in \mathsf{2}^* \mid a \text{ does not clash with } \langle m_{-}, d_{-}\rangle\}$.
5. K_T is a Kleene tree!

Construction of a Kleene Tree

$K_T = \{a \in \mathsf{2}^* \mid a \text{ does not clash with } \langle m_-, d_- \rangle\}$

K_T is a Kleene tree:

1. Clearly, decidable, inhabited, prefix-closed.
2. Unbounded: define $[a_0, \ldots, a_n]$ by

$$a_j = \begin{cases} d_k & \text{if } j = m_k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $[a_0, \ldots, a_n]$ does not clash with $\langle m_-, d_- \rangle$.
3. Every path $\alpha \in \mathsf{2}^{\mathbb{N}}$ exits T_K: α and s are inconsistent, hence prefixes of α clash with $\langle m_-, d_- \rangle$ eventually.

Note: there is an enumeration $\ell : \mathbb{N} \to \mathsf{2}^*$ without repetitions of the leaves of T_K.

Cantor space and Baire space

The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are complete separable metric spaces, with metric (for both spaces)

$$d(\alpha, \beta) = 2^{-\min\{k \in \mathbb{N} \mid \alpha_k \neq \beta_k\}}.$$

Theorem

$2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are homeomorphic as metric spaces.*

Proof.

The homeomorphism $h : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by

$$h(\alpha) = \ell(\alpha_0)\ell(\alpha_1)\ell(\alpha_2)\cdots$$

□

Computing $2^{2^{\mathbb{N}}}$

$2^{2^{\mathbb{N}}}$ is the set of decidable subsets of decidable subsets.

$$2^{2^{\mathbb{N}}} = 2^{\mathbb{N}^{\mathbb{N}}} = 2^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} = (2^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} .$$

Remark: in other models of computability, such as **Equ**, we have $2^{\mathbb{N}} \ncong \mathbb{N}^{\mathbb{N}}$ and $2^{2^{\mathbb{N}}} = \mathbb{N}$.

Local non-compactness of $\mathbb{R}$

- The "middle-thirds" embedding $i : \mathsf{2}^{\mathbb{N}} \to [0, 1]$, $i(\alpha) = \sum_{k=0}^{\infty} \frac{2\alpha_k}{3^{k+1}}$.
- The image $C = \mathrm{im}(i)$ is a closed located subset of $[0, 1]$.
- The map $i \circ h : \mathbb{N}^{\mathbb{N}} \to [0, 1]$ embeds $\mathbb{N}^{\mathbb{N}}$ as a closed located subset $C \subseteq [0, 1]$.

Theorem (Specker sequence)

There exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $[0, 1]$ without accumulation point.

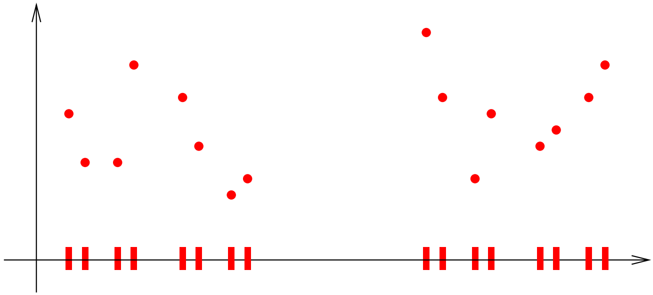
Proof.

The sequence $b_n = \lambda k.\, n$, is without accumulation point in $\mathbb{N}^{\mathbb{N}}$. Define $a_n = i(h(b_n))$. Then a_n is without accumulation point in C. Because C is closed and located, a_n is without accumulation point in $[0, 1]$. □

Extending a continuous map $C \to \mathbb{R}$

Theorem

Every continuous $g : C \to \mathbb{R}$ can be extended to a continuous $\overline{g} : [0,1] \to \mathbb{R}$.



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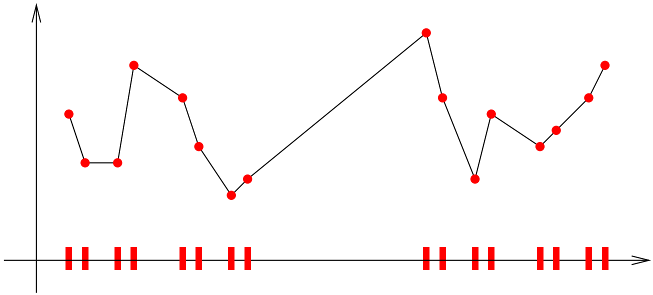

Proof: exercise.

Unbounded continuous $f : [0, 1] \to \mathbb{R}$

Theorem

There exists an unbounded continuous map $[0, 1] \to \mathbb{R}$.

Proof.

- The map $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$, $g : \alpha \mapsto \alpha_0$ is unbounded and continuous.
- The map $g \circ h^{-1} \circ i^{-1} : C \to \mathbb{R}$ is unbounded and continuous.
- Extend $g \circ h^{-1} \circ i^{-1}$ to a continuous $f : [0, 1] \to \mathbb{R}$. It is still unbounded.

$\square$

Conclusion

- The theme: as logicians, we should look for *elegant* presentations of theories we study. They can lead to new intuitions (and destroy old ones).
- These slides, and more, at `math.andrej.com`.
- Further reading:
  - P. Odifreddi: *Classical Recursion Theory*
  - E. Bishop & D. Bridges: *Foundations of Constructive Analysis*
  - D. Bridges & F. Richman: *Varieties of Constructive Mathematics*