

Synthetic Computability (Computability Theory without Computers)

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What is “synthetic” mathematics?

- ▶ Suppose we want to study mathematical structures forming a category \mathcal{C} , such as:
 - ▶ smooth manifolds and differentiable maps
 - ▶ topological spaces and continuous maps
 - ▶ computable sets and computable maps
- ▶ **Classical approach:** structures are *sets equipped with extra structure*, morphisms *preserve the structure*.
- ▶ **Synthetic approach:** embed \mathcal{C} in a suitable *mathematical universe* \mathcal{E} (a model of intuitionistic set theory) and view structures as *ordinary sets* and morphisms as *ordinary maps* inside \mathcal{E} .
- ▶ The synthetic approach is unusual because:
 - (a) construction of a suitable \mathcal{E} may be technically involved,
 - (b) unusual axioms and logic may be valid in \mathcal{E} .
- ▶ But we can forget (a) once it has been done.
- ▶ As logicians, we are delighted about (b) because it takes us to exciting new worlds of mathematics.

A synthetic universe for computability theory

- ▶ M. Hyland's *effective topos* \mathbf{Eff} is the mathematical universe suitable for computability theory.
- ▶ In \mathbf{Eff} all objects and morphisms are equipped with computability structure.
- ▶ We need not know how \mathbf{Eff} is built—we just use the logic and axioms which are valid in it.
- ▶ Our task is to see *what mathematics is like for those who live inside* \mathbf{Eff} .
- ▶ Other synthetic worlds:
 - ▶ Synthetic Differential Geometry [Lawvere & Kock]
 - ▶ Synthetic Topology [Taylor, Escardó]

External and internal view

Comparison of concepts as viewed by us (externally) and by mathematicians inside Eff (internally):

Symbol	External	Internal
\mathbb{N}	natural numbers	natural numbers
\mathbb{R}	<i>computable</i> reals	<i>all</i> reals
$f : \mathbb{N} \rightarrow \mathbb{N}$	<i>computable</i> map	<i>any</i> map
$e : \mathbb{N} \rightarrow A$	<i>computable</i> enumeration of A	<i>any</i> enumeration of A
$\{\text{true}, \text{false}\}$	truth values	decidable truth values
Ω	truth values of Eff	truth values
$\forall x$	for all <i>computable</i> x	for all x
$\exists x$	there exists <i>computable</i> x	there exists x
$P \vee \neg P$	decision procedure for P	P or not P

Translating between external and internal views

- ▶ There is a systematic way to translate any internal statement to an external one, namely the *interpretation of intuitionistic logic and set theory* in \mathbf{Eff} .
- ▶ But we will not do that. Instead we shall speak and think like mathematicians inside \mathbf{Eff} .

Related Work

- ▶ Friedman [1971], axiomatizes coding and universal functions
- ▶ Moschovakis [1971] & Fenstad [1974], axiomatize computations and subcomputations
- ▶ Hyland [1982], effective topos
- ▶ Richman [1984], an axiom for effective enumerability of partial functions, extended in Bridges & Richman [1987]
- ▶ We shall follow Richman [1984] in style, and borrow ideas from Rosolini [1986], Berger [1983], and Spreen [1998].

Outline

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Basic Computability

Theorems for Free

Axiom of Enumerability

Further Results

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Intuitionistic logic

- ▶ Law of Excluded Middle:

$$\forall p \in \Omega . (p \vee \neg p)$$

“For every proposition p , p or not p .”

- ▶ Accepted in classical mathematics (99.9% of mathematicians and falling).
- ▶ In intuitionistic mathematics it can only be used in special cases, when p is *decidable*.

Axiom of Choice

- ▶ Axiom of Choice:

For every family $\{A_i\}_{i \in I}$ of non-empty sets there is a choice function $f : I \rightarrow \bigcup_{i \in I} A_i$, such that $f(i) \in A_i$ for all $i \in I$.

- ▶ Logical formulation of Axiom of Choice: every total relation has a choice function,

$$(\forall x \in A . \exists y \in B . R(x, y)) \implies \exists f \in B^A . \forall x \in A . R(x, f(x)).$$

- ▶ Axiom of Choice \implies Excluded Middle.
- ▶ Constructive mathematics: only *Countable Choice*, which is the case $I = A = \mathbb{N}$.

Basic sets and constructions

- ▶ Basic sets:

$$\emptyset, \quad \mathbf{1} = \{*\}, \quad \mathbb{N} = \{0, 1, 2, \dots\}$$

- ▶ Set operations:

$$A \times B, \quad A + B, \quad B^A = A \rightarrow B, \quad \{x \in A \mid p(x)\}, \quad \mathcal{P}A$$

- ▶ We say that A is
 - ▶ *non-empty* if $\neg \forall x \in A. \perp$,
 - ▶ *inhabited* if $\exists x \in A. \top$.

Some interesting sets

- ▶ The set of truth values:

$$\Omega = \mathcal{P}1$$

$$\text{truth } \top = 1, \quad \text{falsehood } \perp = \emptyset$$

- ▶ The set of *decidable* truth values:

$$\mathbf{2} = \{0, 1\} = \{p \in \Omega \mid p \vee \neg p\},$$

where we write $1 = \top$ and $0 = \perp$.

- ▶ The set of *classical* truth values:

$$\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p = p\}.$$

- ▶ $\mathbf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$.

Decidable and classical sets

- ▶ A subset $S \subseteq A$ is equivalently given by its characteristic map

$$\chi_S : A \rightarrow \Omega, \quad \chi_S(x) = (x \in S).$$

- ▶ A subset $S \subseteq A$ is *decidable* if $\chi_S : A \rightarrow \mathbf{2}$, equivalently

$$\forall x \in A. (x \in S \vee x \notin S) .$$

- ▶ A subset $S \subseteq A$ is *classical* if $\chi_S : A \rightarrow \Omega_{\neg\neg}$, equivalently

$$\forall x \in A. (\neg(x \notin S) \implies x \in S) .$$

Enumerable & finite sets

- ▶ A is *finite* if there exist $n \in \mathbb{N}$ and a surjection

$$e : \{1, \dots, n\} \twoheadrightarrow A,$$

called a *listing* of A . An element may be listed more than once.

- ▶ A is *enumerable (countable)* if there exists a surjection

$$e : \mathbb{N} \twoheadrightarrow A,$$

called an *enumeration* of A . For inhabited A we may take $e : \mathbb{N} \twoheadrightarrow A$.

- ▶ A is *infinite* if there exists an injective $a : \mathbb{N} \hookrightarrow A$.

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Lawvere \rightarrow Cantor

Theorem (Lawvere)

If $e : A \rightarrow B^A$ is surjective then B has the fixed point property: for every $f : B \rightarrow B$ there is $x_0 \in B$ such that $f(x_0) = x_0$.

Proof.

Given $f : B \rightarrow B$, define $g(y) = f(e(y)(y))$. Because e is surjective there is $x \in A$ such that $e(x) = g$. Then $e(x)(x) = f(e(x)(x))$, so $x_0 = e(x)(x)$ is a fixed point of f . \square

Corollary (Cantor)

There is no surjection $e : A \rightarrow \mathcal{P}A$.

Proof.

$\mathcal{P}A = \Omega^A$ and negation $\neg : \Omega \rightarrow \Omega$ does not have a fixed point. \square

Non-enumerability of Cantor and Baire space

Are there any sets which are *not* enumerable?

Yes, for example $\mathcal{P}\mathbb{N}$, and also:

Corollary

$2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are not enumerable.

Proof.

2 and \mathbb{N} do not have the fixed-point property. □

We have proved our first synthetic theorem:

Theorem (external translation of above corollary)

The set of recursive sets and the set of total recursive functions cannot be computably enumerated.

Projection Theorem

Recall: the *projection* of $S \subseteq A \times B$ is the set

$$\{x \in A \mid \exists y \in B. \langle x, y \rangle \in S\}.$$

Theorem (Projection)

A subset of \mathbb{N} is enumerable iff it is the projection of a decidable subset of $\mathbb{N} \times \mathbb{N}$.

Proof.

If A is enumerated by $e : \mathbb{N} \rightarrow 1 + A$ then A is the projection of the *graph* of e , which is the set $\{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \mid m = e(n)\}$.

If A is the projection of $B \subseteq \mathbb{N} \times \mathbb{N}$, define $e : \mathbb{N} \times \mathbb{N} \rightarrow 1 + A$ by

$$e\langle m, n \rangle = \text{if } \langle m, n \rangle \in B \text{ then } m \text{ else } \star.$$



Semidecidable sets

- ▶ A *semidecidable truth value* $p \in \Omega$ is one that is equivalent to

$$\exists n \in \mathbb{N} . d(n)$$

for some $d : \mathbb{N} \rightarrow \mathbf{2}$.

- ▶ The set of semidecidable truth values:

$$\Sigma = \{p \in \Omega \mid \exists d \in \mathbf{2}^{\mathbb{N}} . (p \iff \exists n \in \mathbb{N} . d(n))\} .$$

This is Rosolini's *dominance*.

- ▶ $\mathbf{2} \subseteq \Sigma \subseteq \Omega$.
- ▶ A subset $S \subseteq \mathbb{N}$ is *semidecidable* if $\chi_S : A \rightarrow \Sigma$.

Semidecidable subsets of \mathbb{N}

Theorem

The enumerable subsets of \mathbb{N} are the semidecidable subsets of \mathbb{N} .

Proof.

By Projection Theorem, an enumerable $A \subseteq \mathbb{N}$ is the projection of a decidable $B \subseteq \mathbb{N} \times \mathbb{N}$. Then $n \in A$ iff $\exists m \in \mathbb{N} . \langle n, m \rangle \in B$. Conversely, if $A \in \Sigma^{\mathbb{N}}$, by Number Choice there is $d : \mathbb{N} \times \mathbb{N} \rightarrow 2$ such that $n \in A$ iff $\exists m \in \mathbb{N} . d(m, n)$. □

The enumerable subsets of \mathbb{N} :

$$\mathcal{E} = \Sigma^{\mathbb{N}} .$$

Partial functions

- ▶ A partial function $f : A \rightharpoonup B$ is a function $f : A' \rightarrow B$ defined on a subset $A' \subseteq A$, called the *domain* of f .
- ▶ Write $f(x)\downarrow$ when $x \in A'$ and $f(x)$ is defined.
- ▶ We can view $f : A \rightharpoonup B$ as an ordinary function $f' : A \rightarrow \tilde{B}$ where

$$\tilde{B} = \{s \in \mathcal{P}B \mid \text{every two elements of } s \text{ are equal}\},$$
$$f'(x) = \{y \in B \mid f(x)\downarrow \wedge y = f(x)\}.$$

Then

$$f'(x) = \{y\} \iff f(x)\downarrow \wedge f(x) = y,$$
$$f'(x) = \emptyset \iff f(x) \text{ undefined.}$$

Σ -partial functions

The *graph* of $f : \mathbb{N} \rightarrow \mathbb{N}$ is the set

$$\{\langle x, y \rangle \in A \times B \mid f(x) \downarrow \wedge y = f(x)\}.$$

When does a partial function $f : \mathbb{N} \rightarrow \mathbb{N}$ have an enumerable graph?

Proposition

$f : \mathbb{N} \rightarrow \tilde{\mathbb{N}}$ has an enumerable graph iff $f(n) \downarrow \in \Sigma$ for all $n \in \mathbb{N}$.

Define the *lifting* operation

$$A_{\perp} = \{s \in \tilde{A} \mid (\exists x \in A. x \in s) \in \Sigma\}.$$

For $f : A \rightarrow B$ define $f_{\perp} : A_{\perp} \rightarrow B_{\perp}$ to be

$$f_{\perp}(s) = \{f(x) \mid x \in s\}.$$

A Σ -*partial function* is a function $f : A \rightarrow B_{\perp}$.

Domains of Σ -partial functions

Proposition

A subset is semidecidable iff it is the domain of a Σ -partial function.

Proof.

A semidecidable subset $S \in \Sigma^A$ is the domain of its characteristic map $\chi_S : A \rightarrow \Sigma = \mathbf{1}_\perp$.

If $f : A \rightarrow B_\perp$ is Σ -partial then its domain is the set $\{x \in A \mid f(x) \downarrow\}$, which is obviously semidecidable. □

Theorem (External translation)

A set is semidecidable iff it is the domain of a partial recursive map.

The Single-Value Theorem

A *selection* for $R \subseteq A \times B$ is a partial map $f : A \rightarrow B$ such that, for every $x \in A$,

$$(\exists y \in B . R(x, y)) \implies f(x) \downarrow \wedge R(x, f(x)) .$$

This is like a choice function, except it only chooses when there is something to choose from.

Theorem (Single Value Theorem)

Every semidecidable relation $R \in \Sigma^{\mathbb{N} \times \mathbb{N}}$ has a Σ -partial selection.

Axiom of Enumerability

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

Let $\mathbf{W} : \mathbb{N} \rightarrow \mathcal{E}$ be an enumeration.

Proposition

Σ and \mathcal{E} have the fixed-point property.

Proof.

By Lawvere, $\mathbf{W} : \mathbb{N} \rightarrow \mathcal{E} = \Sigma^{\mathbb{N}} \cong \Sigma^{\mathbb{N} \times \mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

The Law of Excluded Middle Fails

The Law of Excluded Middle says $2 = \Omega$.

Corollary

The Law of Excluded Middle is false.

Proof.

Among the sets $2 \subseteq \Sigma \subseteq \Omega$ only the middle one has the fixed-point property, so $2 \neq \Sigma \neq \Omega$. □

Enumerability of $\mathbb{N} \rightarrow \mathbb{N}_\perp$

Proposition

$\mathbb{N} \rightarrow \mathbb{N}_\perp$ is enumerable.

Proof.

Let $V : \mathbb{N} \rightarrow \Sigma^{\mathbb{N} \times \mathbb{N}}$ be an enumeration. By Single-Value Theorem and Number Choice, there is $\varphi : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N}_\perp)$ such that φ_n is a selection of V_n . The map φ is surjective, as every $f : \mathbb{N} \rightarrow \mathbb{N}_\perp$ is the only selection of its graph. □

Corollary (Formal Church's Thesis)

$\mathbb{N}^{\mathbb{N}}$ is subenumerable (because $\mathbb{N}^{\mathbb{N}} \subseteq \mathbb{N}_\perp^{\mathbb{N}}$).

In other words, $\forall f \in \mathbb{N}^{\mathbb{N}} . \exists n \in \mathbb{N} . f = \varphi_n$.

End of Part I

Walk around and rest your
brain for 10 minutes.

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The Topological View

- ▶ The topological view:

semidecidable subsets = open subsets .

- ▶ Σ is the *Sierpinski space*: the space on two points \perp, \top with $\{\top\}$ open and $\{\perp\}$ closed.
- ▶ The *topology of A* is Σ^A .

Continuous maps

Theorem

All functions are continuous.

Proof.

Given any $f : A \rightarrow B$ and $U \in \Sigma^B$, the set $f^{-1}(U)$ is open because it is classified by $U \circ f : A \rightarrow \Sigma$. □

Topological Exterior and Creative Sets

- ▶ The *exterior* of an open set is the largest open set disjoint from it.
- ▶ An open set $U \in \Sigma^A$ is *creative* if it is without exterior: every $V \in \Sigma^A$ disjoint from U can be enlarged and still be disjoint from U .

Theorem

There exists a creative subset of \mathbb{N} .

Proof.

The familiar $K = \{n \in \mathbb{N} \mid n \in \mathbf{W}_n\}$ is creative. Given any $V \in \mathcal{E}$ with $V = \mathbf{W}_k$ and $K \cap V = \emptyset$, we have $k \notin V$ and $k \notin K$, so $V' = V \cup \{k\}$ is larger and still disjoint from K . □

Immune and Simple Sets

- ▶ A set is *immune* if it is neither finite nor infinite.
- ▶ A set is *simple* if it is open and its complement is immune.

Theorem

There exists a (closed) immune subset of \mathbb{N} .

Proof.

Following Post, consider $P = \{\langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \wedge n \in W_m\}$, and let $f : \mathbb{N} \rightarrow \mathbb{N}_\perp$ be a selection for P . Then

$S = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}. f(m) = n\}$ is the complement of the set we are looking for.

Because $f(m) > 2m$ the set $\mathbb{N} \setminus S$ cannot be finite.

For any infinite enumerable set $U \subseteq \mathbb{N} \setminus S$ with $U = W_m$, we have $f(m) \downarrow, f(m) \in W_m = U$, and $f(m) \in S$, hence U is not contained in $\mathbb{N} \setminus S$. □

Markov Principle

- ▶ If a binary sequence $a \in 2^{\mathbb{N}}$ is not constantly 0, does it contain a 1?
- ▶ For $p \in \Sigma$, does $p \neq \perp$ imply $p = \top$?
- ▶ Is $\Sigma \subseteq \Omega_{\neg\neg}$?

Axiom (Markov Principle)

A binary sequence which is not constantly 0 contains a 1.

Post's Theorem

Theorem

For all $p \in \Omega$,

$$p \in \mathbf{2} \iff p \in \Sigma \wedge \neg p \in \Sigma .$$

Proof.

\Rightarrow If $p \in \mathbf{2}$ then $\neg p \in \mathbf{2}$, therefore $p, \neg p \in \mathbf{2} \subseteq \Sigma$.

\Leftarrow If $p \in \Sigma$ and $\neg p \in \Sigma$ then $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore

$$p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\perp = \top ,$$

as required.



End of Part II

Go out,
count the ducks,
come back.

Focal sets

- ▶ A *focal set* is a set A together with a map $\epsilon_A : A_{\perp} \rightarrow A$ such that $\epsilon_A(\{x\}) = x$ for all $x \in A$:

$$\begin{array}{ccc} A & \xrightarrow{\{-\}} & A_{\perp} \\ & \searrow & \downarrow \epsilon_A \\ & & A \end{array}$$

The *focus* of A is $\perp_A = \epsilon_A(\perp)$.

- ▶ A lifted set A_{\perp} is always focal (because lifting is a monad whose unit is $\{-\}$).

Enumerable focal sets

- ▶ Enumerable focal sets, known as *Eršov complete sets*, have good properties.
- ▶ A *flat domain* A_{\perp} is focal. It is enumerable if A is decidable and enumerable.
- ▶ If A is enumerable and focal then so is $A^{\mathbb{N}}$:

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_{\perp}^{\mathbb{N}} \xrightarrow{e_{\perp}^{\mathbb{N}}} A_{\perp}^{\mathbb{N}} \xrightarrow{\epsilon_A^{\mathbb{N}}} A^{\mathbb{N}}$$

- ▶ Some enumerable focal sets are

$$\Sigma^{\mathbb{N}}, \quad \mathbf{2}_{\perp}^{\mathbb{N}}, \quad \mathbb{N}_{\perp}^{\mathbb{N}}.$$

Multi-valued functions

- ▶ A *multi-valued function* $f : A \rightrightarrows B$ is a function $f : A \rightarrow \mathcal{P}B$ such that $f(x)$ is inhabited for all $x \in A$.
- ▶ This is equivalent to having a *total relation* $R \subseteq A \times B$. The connection between f and R is

$$f(x) = \{y \in B \mid R(x, y)\}$$
$$R(x, y) \iff y \in f(x) .$$

- ▶ A *fixed point* of $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Recursion Theorem

Theorem (Recursion Theorem)

Every $f : A \rightrightarrows A$ on enumerable focal A has a fixed point.

Proof.

Let $e : \mathbb{N} \rightarrow A$ be an enumeration, and $\epsilon : A_{\perp} \rightarrow A$ a focal map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $e(m) \in f(e(k))$. By Number Choice there is a map $c : \mathbb{N} \rightarrow \mathbb{N}$ such that $e(c(k)) \in f(e(k))$ for every $k \in \mathbb{N}$. It suffices to find k such that $e(c(k)) = e(k)$ since then $x = e(k)$ is a fixed point for f .

For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\epsilon(e_{\perp}(c_{\perp}(\varphi_m(m)))) = e(n)$. By Number Choice there is $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\epsilon(e_{\perp}(c_{\perp}(\varphi_m(m)))) = e(g(m))$ for every $m \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $g = \varphi_j$. Let $k = g(j)$. Then

$$e(k) = e(g(j)) = \epsilon(e_{\perp}(c_{\perp}(\varphi_j(j)))) = e(c(g(j))) = e(c(k)) .$$



Can someone make this proof more beautiful?

Classical Recursion Theorem

Corollary (Classical Recursion Theorem)

For every $f : \mathbb{N} \rightarrow \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\varphi_{f(n)} = \varphi_n$.

Proof.

In Recursion Theorem, take the enumerable focal set $A = \mathbb{N}_{\perp}^{\mathbb{N}}$ and the multi-valued function

$$F(g) = \{h \in \mathbb{N}_{\perp}^{\mathbb{N}} \mid \exists n \in \mathbb{N} . g = \varphi_n \wedge h = \varphi_{f(n)}\} .$$

There is g such that $g \in F(g)$. Thus there exists $n \in \mathbb{N}$ such that $\varphi_n = g = h = \varphi_{f(n)}$. □

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End of Part III

Please vote:

1. End now, please!
2. Go on until 7:30 without a pause.

Plotkin's Domain $2_{\perp}^{\mathbb{N}}$

- ▶ In a *partially ordered set* $(P, \leq$ we say that x and y are *incomparable* if $x \not\leq y$ and $y \not\leq x$.
- ▶ The set of Σ -partial binary functions $2_{\perp}^{\mathbb{N}}$ is a partially ordered:

$$f \leq g \iff \forall n \in \mathbb{N}. (f(n) \downarrow \implies g(n) \downarrow \wedge f(n) = g(n))$$

It is called *Plotkin's universal domain* in domain theory.

- ▶ Suppose P is a partially ordered set and $x \in P$. Is there a maximal element of P above x ?

Inseparable sets

Theorem

There exists an element of $2_{\perp}^{\mathbb{N}}$ that is inconsistent with every maximal element of $2_{\perp}^{\mathbb{N}}$.

Proof.

Because 2_{\perp} is focal and enumerable, $2_{\perp}^{\mathbb{N}}$ is as well. Let $\psi : \mathbb{N} \rightarrow 2_{\perp}^{\mathbb{N}}$ be an enumeration, and let $t : 2_{\perp} \rightarrow 2_{\perp}$ be the isomorphism $t(x) = \neg_{\perp} x$ which exchanges 0 and 1, and fixes \perp . Consider $a \in 2_{\perp}^{\mathbb{N}}$ defined by $a(n) = t(\psi_n(n))$. If $b \in 2_{\perp}^{\mathbb{N}}$ is maximal with $b = \psi_k$, then $a(k) = \neg_{\perp} \psi_k(k) = \neg_{\perp} b(k)$. Because $a(k)$ and $b(k)$ are both total and different they are inconsistent. Hence a and b are inconsistent. □

Binary Trees

- ▶ Let 2^* be the set of finite binary sequences, with prefix-ordering \preceq .
- ▶ The *length* of $[a_0, \dots, a_{n-1}] \in 2^*$ is $|a| = n$.
- ▶ A *tree* $T \subseteq 2^*$ is an inhabited prefix-closed subset.
- ▶ A *Kleene tree* T_K is a tree such that:
 1. T_K is decidable (as a subset of 2^*),
 2. T_K is unbounded: $\forall k \in \mathbb{N}. \exists a \in T_K. |a| \geq k$,
 3. every infinite path exits T_K :

$$\forall \alpha \in 2^{\mathbb{N}}. \exists n \in \mathbb{N}. [\alpha_0, \dots, \alpha_n] \notin T_K .$$

Construction of a Kleene Tree

1. Recall an enumeration $\psi : \mathbb{N} \rightarrow 2_{\perp}^{\mathbb{N}}$ and $s(n) = \neg_{\perp} \psi_n(n)$ which is inconsistent with every $\alpha \in 2^{\mathbb{N}}$.
2. Let $\langle m_{-}, d_{-} \rangle : \mathbb{N} \rightarrow \mathbb{N} \times \mathbf{2}$ be an enumeration of the graph of s , i.e., $s(m_k) = d_k$ for all $k \in \mathbb{N}$ and we enumerate all such pairs.
3. Given $a = [a_0, \dots, a_n] \in 2^*$, say that a *clashes with* $\langle m_{-}, d_{-} \rangle$, if there is $k \leq n$ such that $m_k \leq n$ and $a_{m_k} \neq d_k$.
4. Define $K_T = \{a \in 2^* \mid a \text{ does not clash with } \langle m_{-}, d_{-} \rangle\}$.
5. K_T is a Kleene tree!

Construction of a Kleene Tree

$$K_T = \{a \in 2^* \mid a \text{ does not clash with } \langle m_-, d_- \rangle\}$$

K_T is a Kleene tree:

1. Clearly, decidable, inhabited, prefix-closed.
2. Unbounded: define $[a_0, \dots, a_n]$ by

$$a_j = \begin{cases} d_k & \text{if } j = m_k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[a_0, \dots, a_n]$ does not clash with $\langle m_-, d_- \rangle$.

3. Every path $\alpha \in 2^{\mathbb{N}}$ exits T_K : α and s are inconsistent, hence prefixes of α clash with $\langle m_-, d_- \rangle$ eventually.

Note: there is an enumeration $\ell : \mathbb{N} \rightarrow 2^*$ without repetitions of the leaves of T_K .

Cantor space and Baire space

The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are complete separable metric spaces, with metric (for both spaces)

$$d(\alpha, \beta) = 2^{-\min\{k \in \mathbb{N} \mid \alpha_k \neq \beta_k\}} .$$

Theorem

$2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are homeomorphic as metric spaces.

Proof.

The homeomorphism $h : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ is defined by

$$h(\alpha) = \ell(\alpha_0)\ell(\alpha_1)\ell(\alpha_2)\cdots$$



Computing $2^{2^{\mathbb{N}}}$

$2^{2^{\mathbb{N}}}$ is the set of decidable subsets of decidable subsets.

$$2^{2^{\mathbb{N}}} = 2^{\mathbb{N}^{\mathbb{N}}} = 2^{\mathbb{N} \times \mathbb{N}^{\mathbb{N}}} = (2^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N} \times \mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}.$$

Remark: in other models of computability, such as **Equ**, we have $2^{\mathbb{N}} \not\cong \mathbb{N}^{\mathbb{N}}$ and $2^{2^{\mathbb{N}}} = \mathbb{N}$.

Local non-compactness of \mathbb{R}

- ▶ The “middle-thirds” embedding $i : 2^{\mathbb{N}} \rightarrow [0, 1]$,
$$i(\alpha) = \sum_{k=0}^{\infty} \frac{2\alpha_k}{3^{k+1}}.$$
- ▶ The image $C = \text{im}(i)$ is a closed located subset of $[0, 1]$.
- ▶ The map $i \circ h : \mathbb{N}^{\mathbb{N}} \rightarrow [0, 1]$ embeds $\mathbb{N}^{\mathbb{N}}$ as a closed located subset $C \subseteq [0, 1]$.

Theorem (Specker sequence)

There exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $[0, 1]$ without accumulation point.

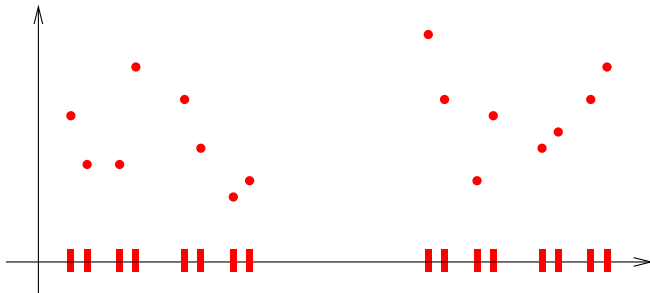
Proof.

The sequence $b_n = \lambda k. n$, is without accumulation point in $\mathbb{N}^{\mathbb{N}}$. Define $a_n = i(h(b_n))$. Then a_n is without accumulation point in C . Because C is closed and located, a_n is without accumulation point in $[0, 1]$. \square

Extending a continuous map $C \rightarrow \mathbb{R}$

Theorem

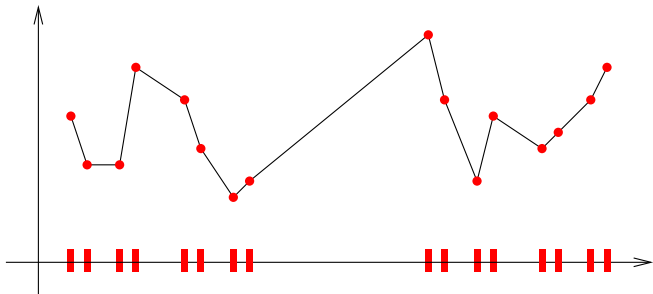
Every continuous $g : C \rightarrow \mathbb{R}$ can be extended to a continuous $\bar{g} : [0, 1] \rightarrow \mathbb{R}$.



Extending a continuous map $C \rightarrow \mathbb{R}$

Theorem

Every continuous $g : C \rightarrow \mathbb{R}$ can be extended to a continuous $\bar{g} : [0, 1] \rightarrow \mathbb{R}$.



Proof: exercise.

Unbounded continuous $f : [0, 1] \rightarrow \mathbb{R}$

Theorem

There exists an unbounded continuous map $[0, 1] \rightarrow \mathbb{R}$.

Proof.

- ▶ The map $g : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$, $g : \alpha \mapsto \alpha_0$ is unbounded and continuous.
- ▶ The map $g \circ h^{-1} \circ i^{-1} : C \rightarrow \mathbb{R}$ is unbounded and continuous.
- ▶ Extend $g \circ h^{-1} \circ i^{-1}$ to a continuous $f : [0, 1] \rightarrow \mathbb{R}$. It is still unbounded.



Conclusion

- ▶ The theme: as logicians, we should look for *elegant* presentations of theories we study. They can lead to new intuitions (and destroy old ones).
- ▶ These slides, and more, at math.andrej.com.
- ▶ Further reading:
 - ▶ P. Odifreddi: *Classical Recursion Theory*
 - ▶ E. Bishop & D. Bridges: *Foundations of Constructive Analysis*
 - ▶ D. Bridges & F. Richman: *Varieties of Constructive Mathematics*